ODE 5-12

Goal:
- Finish undetermined coefficients method for non-homogeneous ODEs
- Fundamental techniques in linear algebra

Eg: Solve the homogeneous ODE (3rd order)

$$y''' + 3y'' + 3y' + y = 0$$

Char. poly. $r^3 + 3r^2 + 3r + 1 = 0$

One root: $-1 = r$ Others: roots of $\frac{p(r)}{r+1} = r^2 + 2r + 1$, or $= (r+1)^2$

$$\begin{array}{r} r^2 + 2r + 1 \\ r+1 \,\big)\, \overline{r^3 + 3r^2 + 3r + 1} \\ \underline{r^3 + r^2} \qquad\qquad \\ 2r^2 + 3r \qquad \\ \underline{2r^2 + 2r} \qquad \\ r + 1 \\ \underline{r + 1} \\ 0 \end{array}$$

Roots: $r = -1, -1, -1$.

General solution: $c_1 \overset{y_1}{e^{-t}} + c_2 \overset{y_2}{te^{-t}} + c_3 \overset{y_3}{t^2 e^{-t}}$ $c_1, c_2, c_3 \in \mathbb{R}$

Verify this is actually the general solution: $\{e^{-t}, te^{-t}, t^2e^{-t}\}$ is a fundamental set of solutions

$$W[y_1, y_2, y_3](t_0) = \det \begin{pmatrix} y_0(t_0) & y_1(t_0) & y_2(t_0) \\ y_0'(t_0) & y_1'(t_0) & y_2'(t_0) \\ y_0''(t_0) & y_1''(t_0) & y_2''(t_0) \end{pmatrix}$$

$t_0 = 0$

good enough because linear

$y_0 = e^{-t} = 1$ $y_1 = te^{-t} = 0$ $y_2 = t^2 e^{-t} = 0$

$y_0' = -e^{-t} = -1$ $y_1' = -te^{-t} + e^{-t} = 1$ $y_2' = 2te^{-t} - t^2e^{-t} = 0$

$y_0'' = e^{-t} = 1$ $y_1'' = -e^{-t} + te^{-t} - e^{-t} = -2$ $y_2'' = 2(-te^{-t} + e^{-t}) - (2te^{-t} - t^2e^{-t}) = 2$

$$W[y_1, y_2, y_3](0) = \det \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 2 \end{pmatrix} = 1 \begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} - 0 \begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 1(1 \cdot 2 - 0(-2)) = 2 \neq 0$$

$W \neq 0$ at $t_0 = 0$ $\therefore$ fundamental set of solutions.

Resonance in forced systems

(Application of undetermined coefficients)

ODE $y'' + m(t)y' + k(t) = \text{Polynomial} + \left\{\begin{matrix}\sin(at) \\ \cos(bt)\end{matrix}\right\} + e^{ct}$

MoUC: Guess $P(t) = At^2 + Bt + C + D\cos(at) + \ldots$

find $A, B, C, D, \ldots$

the general solutions is $\underbrace{c_1 y_1(t) + c_2 y_2(t)}_{\text{complementary}} + \underbrace{P(t)}_{\text{Particular}}$

Eg: (Forced oscilator)

$$y'' + \omega_F^2 y = \cos(\omega_I t)$$

$$\omega_F > 0$$

$\xleftarrow{\text{forcing}}$ $\cos(\omega_I t)$

frequency ω_F

Eg: "spring + mass" = radio reciever

"forcing" = radio wave

Found last time:

complementary sol'n

$$c_1 \cos(\omega_F t) + c_2 \sin(\omega_F t)$$

$$P(t) = A\cos(\omega_I t) + B\sin(\omega_I t)$$

$$A = \frac{1}{\omega_F^2 - \omega_I^2} \cos(\omega_I t) \quad , B = 0$$

only works $\omega_F^2 - \omega_I^2 \neq 0$ $\longleftarrow$ otherwise this ansatz doesn't work

Suppose $\omega_F^2 = \omega_I^2$ $\therefore$ Try a different ansatz

$$P(t) = At\cos(\omega_I t) + Bt\sin(\omega_I t) + C\cos(\omega_I t) + D\sin(\omega_I t)$$

$$\rho'(t) = A\left(-\omega_I t\sin(\omega t) + \cos(\omega_I t)\right) + B\left(\omega_I t\cos(\omega_I t) + \sin(\omega_I t)\right)$$
$$+ -C\omega_I \sin(\omega t) + \omega_I D\cos(\omega_I t)$$

$$\rho''(t) = A(-\omega_I)\left(\omega_I t\cos(\omega_I t) + \sin(\omega_I t)\right) - A\omega_I \sin(\omega_I t) +$$
$$B\omega_I\left(-\omega_I t\sin(\omega_I t) + \cos(\omega_I t)\right) + \omega_I B\cos(\omega_I t) +$$
$$- C\omega_I^2\cos(\omega_I t) - \omega_I^2 D\sin(\omega_I t)$$

	$t\cos$	$t\sin$	$\cos$	$\sin$
ρ	A	B	C	D
ρ'	$B\omega_I$	$-\omega_I A$	$D\omega_I - A$	$B\omega_I - C\omega_I$
ρ''	$-A\omega_I^2$	$-B\omega_I^2$	$-C\omega_I^2 + 2B\omega_I$	$-D\omega_I^2 - 2A\omega_I$
$r(t)$	0	0	1	0
$L[\rho] = \rho'' + \omega_I^2\rho$	0	0	$2B\omega_I$	$-2A\omega_I$

We get solution

$$B = \frac{1}{2\omega_I}, \quad A, C, D = 0$$

Particular solution

$$\rho(t) = \frac{1}{2\omega_I} t\sin(\omega_I t)$$

General solution:

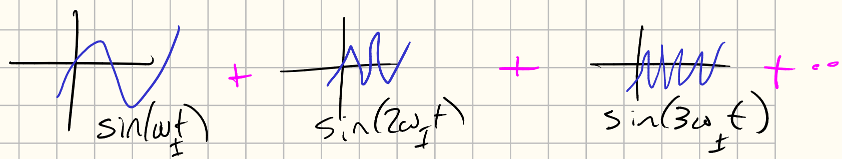
$$c_1\cos(\omega_I t) + c_2\sin(\omega_I t) + \begin{cases} \dfrac{1}{\omega_F^2 - \omega_I^2}\cos(\omega_I t) & \omega_F^2 - \omega_I^2 \neq 0 \\ \dfrac{t}{2\omega_I}\sin(\omega_I t) & \omega_F^2 = \omega_I^2 \end{cases}$$

Bounded as $t \to \infty$ if $\omega_F^2 \neq \omega_I^2$ $\xrightarrow{\text{higher \& higher bounds as } \omega_F^2 \to \omega_I^2}$ $y(t) \to \infty$ as $t \to \infty$ if $\omega_F^2 = \omega_I^2$

Rmk: The same method works for any periodic input.

$$y'' + \omega_F^2 y = \underbrace{\sum_{n=0}^{\infty} a_n \cos(n\omega_I t) + b_n \sin(n\omega_I t)}_{}$$

Any periodic input w/ frequency ω_I has a Fourier series



Fundamental techniques of linear algebra

i) Matrix multiplication

ii) Calculate determinants

iii) Gaussian elimination

inverse of a matrix

Solve systems of linear equations

iv) Use ii) to find eigenvalues of a matrix

v) Use iii) to find eigenvectors of a matrix

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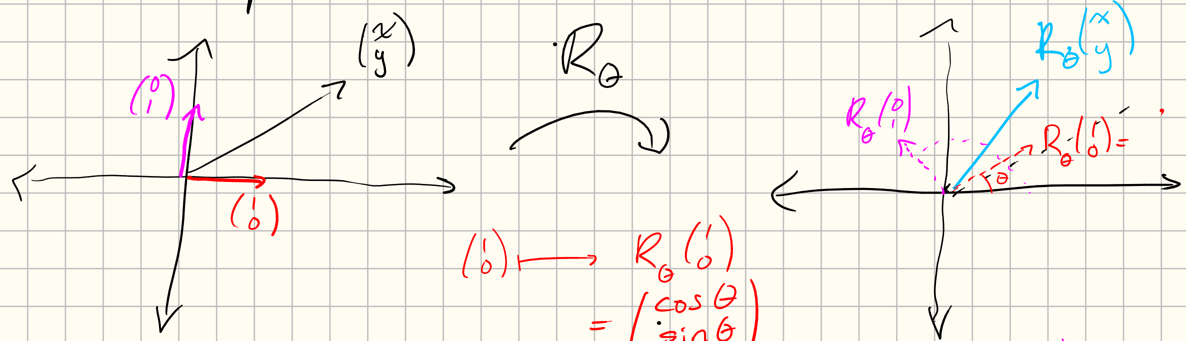
i) Matrix multiplication capture composition of linear maps.

Eg:



$$\binom{1}{0} \longmapsto R_\theta\binom{1}{0} = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

$$\binom{0}{1} \longmapsto R_\theta\binom{0}{1} = \begin{pmatrix}\cos(\theta+\frac{\pi}{2}) \\ \sin(\theta+\frac{\pi}{2})\end{pmatrix} = \begin{pmatrix}-\sin(\theta) \\ \cos(\theta)\end{pmatrix}$$

We can capture this as a matrix

$$R_\theta = \left(c_1 \middle| c_2\right) \qquad \begin{matrix} c_1 = R_\theta\binom{1}{0} \\ c_2 = R_\theta\binom{0}{1}\end{matrix} \qquad R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$

$$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = x R_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y R_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$=: \left(c_1 \,\middle|\, c_2 \right) \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

Trick:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{array}{cc} & \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \end{array}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

Same trick works for $A \cdot B$ for matrices A & B

Eg: Expect

$$R_\theta \cdot R_\phi = R_{\theta+\phi}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{array}{cc} & \begin{pmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{pmatrix} \\ \begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix} & \begin{pmatrix} c\theta c\phi - s\theta s\phi & -c\theta s\phi - s\theta c\phi \\ s\theta c\phi + c\theta s\phi & -\sin\theta s\phi + c\theta c\phi \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$$

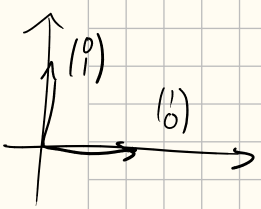
✓

Important!: $AB \neq BA$

Eg:

$$R_\theta F_x = \begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} c\theta & s\theta \\ s\theta & -c\theta \end{pmatrix}$$

$$F_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad F_x R_\theta \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix} = \begin{pmatrix} c\theta & -s\theta \\ -s\theta & -c\theta \end{pmatrix}$$



$R_\theta F_x \neq F_x R_\theta$ (except special value of θ)

ii) Calculate det $\qquad \det(\because) = |\because|$

$$2\times 2: \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$3\times 3: \quad \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$n\times n: \quad \det\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots \\ a_{21} & a_{22} & & \\ & & \ddots & \\ a_{n1} & & & a_{nn} \end{pmatrix} = a_{11}\left|\quad\right| - a_{12}\left|\quad\right| + a_{13}\left|\quad\right| \cdots + (-1)^{n+1}\left|\quad\right|$$

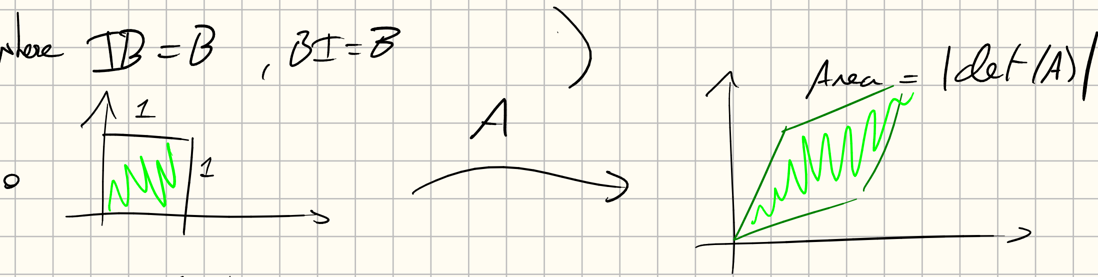
<u>Facts</u>: ∘ $\det(A) \neq 0 \Rightarrow A$ is invertible

(i.e. there is a matrix A^{-1} s.t. $\quad A\cdot A^{-1} = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$A^{-1}A = I$

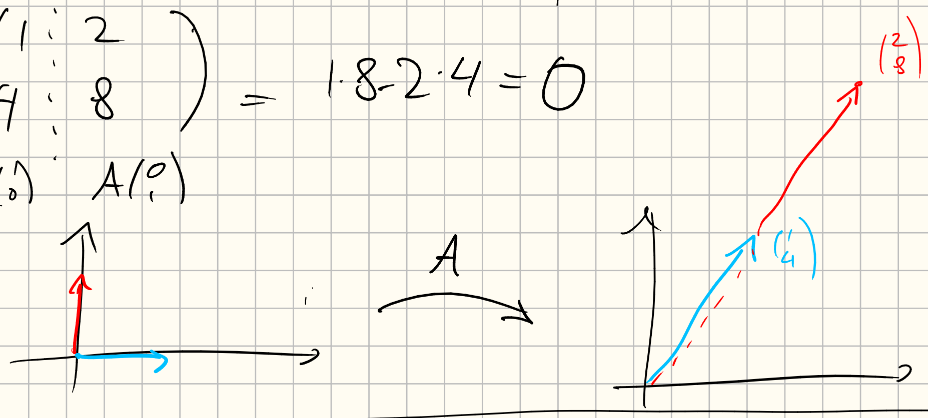
where $IB = B$, $BI = B$)

∘



Eg: $\det\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} = 1\cdot 8 - 2\cdot 4 = 0$

$A\binom{1}{0} \quad A\binom{0}{1}$



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iii) Gaussian elimination $\left(\begin{array}{c} \alpha r_1 \\ \hline r_2 \\ \hline r_3 \end{array}\right)$

$$\left(\begin{array}{c} r_1 \\ \hline r_2 \\ \hline r_3 \end{array}\right) \rightsquigarrow \left(\begin{array}{c} r_2 \\ \hline r_1 \\ \hline r_3 \end{array}\right) \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightsquigarrow \left(\begin{array}{c} r_1 \\ \hline r_2 \\ \hline r_3 - \beta r_1 \end{array}\right)$$

Eg: (use G.E. to solve system of linear equations)

$$x_1 - 2x_2 + 3x_3 = 7$$
$$-x_1 + x_2 - 2x_3 = -5$$
$$2x_1 - x_2 - x_3 = 4$$

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ 4 \end{pmatrix}$$

$$Ax = y$$

(one way $x = A^{-1} y$)

Step 2: Write augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 2 & -1 & -1 & 4 \end{array}\right) \xrightarrow{r_3 \to r_3 - 2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 0 & -5 & -7 & -10 \end{array}\right)$$

$$\xrightarrow{r_2 \to r_2 + r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 3 & 1 & 2 \\ 0 & -5 & -7 & -10 \end{array}\right) \xrightarrow{r_2 \to \frac{1}{3} r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 1 & \frac{1}{3} & 2/3 \\ 0 & -5 & -7 & -10 \end{array}\right)$$

$$\xrightarrow{r_3 \to r_3 + 5r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 1 & 1/3 & 2/3 \\ 0 & 0 & -16/3 & -10 + 10/3 \end{array}\right) \xrightarrow{r_3 \mapsto \frac{-3}{16} r_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 1 & \frac{1}{3} & 2/3 \\ 0 & 0 & 1 & (-10 + 10/3)\frac{-3}{16} \end{array}\right)$$

We have an equivalent eq.

$$x_1 + 2x_2 + 3x_3 = 7$$
$$x_2 + \tfrac{1}{3} x_3 = 2/3$$
$$x_3 = \left(-10 + \tfrac{10}{3}\right)\frac{-3}{16} = \frac{5}{4}$$

$x_3 = 5/4$, $x_2 = \frac{2}{3} - \frac{1}{3}\frac{5}{4}$, $x_1 = 7 - 2x_2 - 3x_3$.

Next time:

iv) eigenvalues: solutions λ to polynomial $\det(A - \lambda I) = 0$

$$\det\left(A - \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda \end{pmatrix}\right) = 0$$

v) eigenvectors (i.e. solutions to $Ax = \lambda x$) where λ is an eigenvalue

Use G.E. to solve $(A - \lambda I)x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$