ODE 5-13

Goal: • Start applying linear algebra to ODEs

Eg: Calculate a matrix inverse:

Given a matrix A we want A^{-1} s.t.

$(AA^{-1})v = v$. Arrange this by finding A^{-1} s.t. $AA^{-1} = I$.

reason: if $I = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ then $I\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = v_1\begin{pmatrix}1\\0\\0\end{pmatrix} + v_2\begin{pmatrix}0\\1\\0\end{pmatrix} + v_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$

Solve for columns of A^{-1} using Gaussian elimination

$A^{-1} = \left(c_1 \middle| c_2 \middle| c_3\right) \qquad AA^{-1} = I = A\left(c_1 \middle| c_2 \middle| c_3\right) = \left(\begin{matrix}1\\0\\0\end{matrix}\middle|\begin{matrix}0\\1\\0\end{matrix}\middle|\begin{matrix}0\\0\\1\end{matrix}\right)$

$\left(Ac_1 \middle| Ac_2 \middle| Ac_3\right) = \cdots$

Solve equations $Ac_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $Ac_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $Ac_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

Notational shorthand:

method

$$\boxed{\left(A \,\middle|\, \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right) \xrightarrow{GE} \left(\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} \,\middle|\, A^{-1}\right)}$$

Eg: Find A^{-1} if $A = \begin{pmatrix}3&2&2\\1&4&1\\2&-4&-1\end{pmatrix}$

$$\left(\begin{matrix}3&2&2\\1&4&1\\2&-4&-1\end{matrix}\middle|\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right) \xrightarrow{r_1 \to \frac{1}{3}r_1} \left(\begin{matrix}1&\frac{2}{3}&\frac{2}{3}\\1&4&1\\2&-4&-1\end{matrix}\middle|\begin{matrix}\frac{1}{3}&0&0\\0&1&0\\0&0&1\end{matrix}\right)$$

$$\xrightarrow{r_2 \to r_2 - r_1} \left(\begin{matrix}1&\frac{2}{3}&\frac{2}{3}\\0&\frac{10}{3}&\frac{1}{3}\\2&-4&-1\end{matrix}\middle|\begin{matrix}\frac{1}{3}&0&0\\-\frac{1}{3}&1&0\\0&0&1\end{matrix}\right) \xrightarrow{r_3 \to r_3 - 2r_1} \left(\begin{matrix}1&\frac{2}{3}&\frac{2}{3}\\0&\frac{10}{3}&\frac{1}{3}\\0&-\frac{16}{3}&-\frac{7}{3}\end{matrix}\middle|\begin{matrix}\frac{1}{3}&0&0\\-\frac{1}{3}&1&0\\-\frac{2}{3}&0&1\end{matrix}\right)$$

$$\xrightarrow{r_2 \to \frac{3}{10}r_2} \left(\begin{matrix}1&\frac{2}{3}&\frac{2}{3}\\0&1&\frac{1}{10}\\0&-\frac{16}{3}&-\frac{7}{3}\end{matrix}\middle|\begin{matrix}\frac{1}{3}&0&0\\-\frac{1}{10}&\frac{3}{10}&0\\-\frac{2}{3}&0&1\end{matrix}\right) \xrightarrow{r_1 \to r_1 - \frac{3}{2}r_2} \left(\begin{matrix}1&0&\frac{31}{60}\\0&1&\frac{1}{10}\\0&-\frac{16}{3}&-\frac{7}{3}\end{matrix}\middle|\begin{matrix}\frac{29}{60}&-\frac{9}{20}&0\\-\frac{1}{10}&\frac{3}{10}&0\\-\frac{2}{3}&0&1\end{matrix}\right)$$

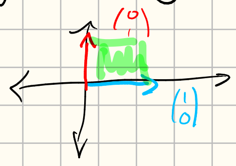
$$\rightsquigarrow \longrightarrow \left(\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\middle| A^{-1}\right) \qquad A^{-1} = \frac{1}{18}\begin{pmatrix}0&6&6\\-3&7&1\\12&-16&-10\end{pmatrix}$$

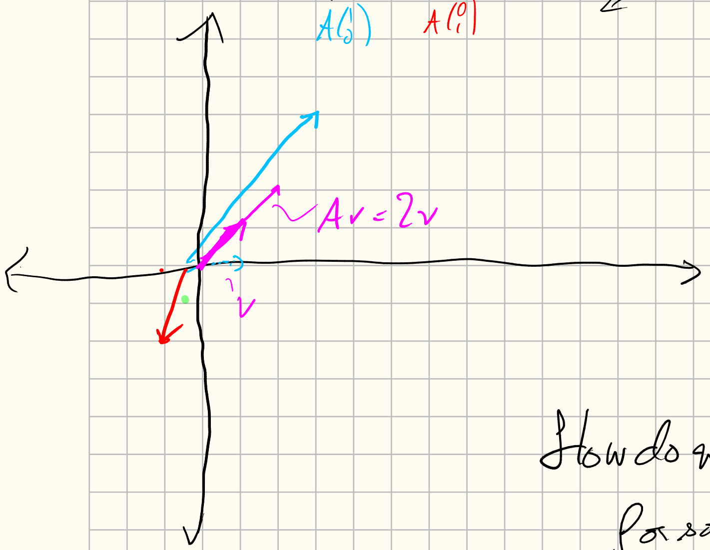
iv)
v) <u>Eigenvectors & eigenvalues</u>

<u>Why?</u> Sometimes when we have a matrix it's a simple geometric operation, we're just using the wrong coordinates

<u>Eg:</u> $A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$

$A\binom{1}{0}$ $\quad A\binom{0}{1}$





What happens to the vector

$v = \binom{1}{1}$?

$$Av = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-1 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

How do we find all v s.t. $Av = \lambda v$ for some λ.

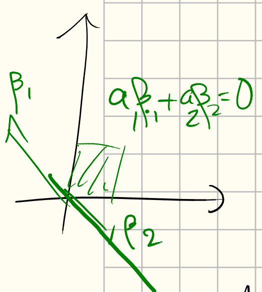
If we knew λ,
$$Av = \lambda v$$
$$Av - \lambda v = 0$$
$$(A - \lambda I)v = 0 \quad \leftarrow \text{can solve using G.E.}$$

<u>Fact:</u> The equation $Bv = 0$ (B is $n \times n$) has a solution $v \neq 0$ if and only if $\det(B) = 0$.

<u>Pf:</u> (otherwise B^{-1} exists $v = B^{-1}(0) = 0$) Other direction follows from geometry: or the fact that $\det(B) = 0 \Rightarrow$ linear relation between columns

i.e. $\exists\, a_1, \dots, a_n$ s.t. if $B = (\beta_1 | \cdots | \beta_n)$ $\det B = 0$

$a_1\beta_1 + \cdots + a_n\beta_n = 0$, Then $v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ $\quad Bv = 0$



Conclusion: $Av = \lambda v$ has solutions $v \neq 0$ if and only if $\det(A - \lambda I) = 0$

<u>Def</u> $\det(A - \lambda I)$ is called the <u>characteristic polynomial</u> of A.

<u>Def:</u> The roots of $\det(A - \lambda I)$ are the <u>eigenvalues</u> λ_i of A.

Def: Eigenvectors ξ_λ with eigenvalue λ are vectors ξ_λ s.t. $A\xi_\lambda = \lambda \xi_\lambda$.

Eg: (iv) v) Find eigenvectors and eigenvalues of

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

char. poly. $= \det(A - \lambda I) = \det\left(\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$

$$= \det\begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix} = (3-\lambda)(-2-\lambda) - (-1)\cdot 4$$

$$= -6 + 2\lambda - 3\lambda + \lambda^2 + 4$$

$$= -2 - \lambda + \lambda^2$$

$$= (\lambda - 2)(\lambda + 1)$$

eigenvalues are $2, -1$ $= \lambda$

Find eigenvectors:

$\lambda = 2$: $\quad (A - 2I)\xi_2 = 0$

$$\left[\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]\xi_2 = 0$$

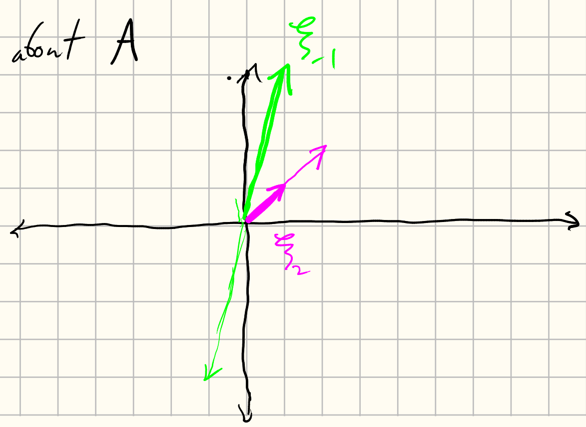
$$\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\xi_2 = 0 \qquad \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

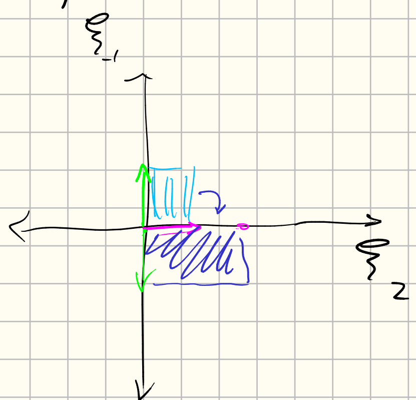
$\lambda = -1$: $\quad [A - (-1)I]\xi_{-1} = 0$

$$\begin{pmatrix} 3+1 & -1 \\ 4 & -2+1 \end{pmatrix}\xi_{-1} = 0$$

$$\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}\xi_{-1} = 0 \qquad \xi_{-1} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Conclusion about A





Eg: Calculate an eigenvalue & corresponding eigenvector of $A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ 2 & -4 & -1 \end{pmatrix}$

char poly: $\det \begin{pmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ 2 & -4 & -1-\lambda \end{pmatrix} =$

$$(3-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -4 & -1-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 2 & -1-\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & 4-\lambda \\ 2 & -4 \end{vmatrix} =$$

$$(3-\lambda)\left[(4-\lambda)(-1-\lambda)+4\right] - 2(-1-\lambda-2) + 2(-4-8+2\lambda) =$$

$$-\lambda^3 + (3+4-1)\lambda^2 + \underbrace{(-12+3+4-4+2+4)}_{11}\lambda + 6$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$\lambda = \underline{1}$ is a root (others using long division)

Find eigenvector ξ_λ by solving

$\underline{\lambda = 1}$: $(A - I)\xi_\lambda = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix}\xi_\lambda = 0 \qquad \text{Notice: } col_1 = col_3$$

so: $\xi_\lambda = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ since $(A-I)\xi_\lambda = col_1 - col_3 = 0$

Why are we doing all of this?

Reason 1: For linear equations, linear changes of coordinates can drastically simplify your equation. (We can also solve some non-linear equations)

Eg: $\dfrac{dy}{dx} = \dfrac{3y - x}{4y - 2x}$ ⇝ becomes a system of equations $\begin{cases} \dfrac{dx_1}{dt} = 3x_1 - x_2 \\ \dfrac{dx_2}{dt} = 4x_1 - 2x_2 \end{cases}$

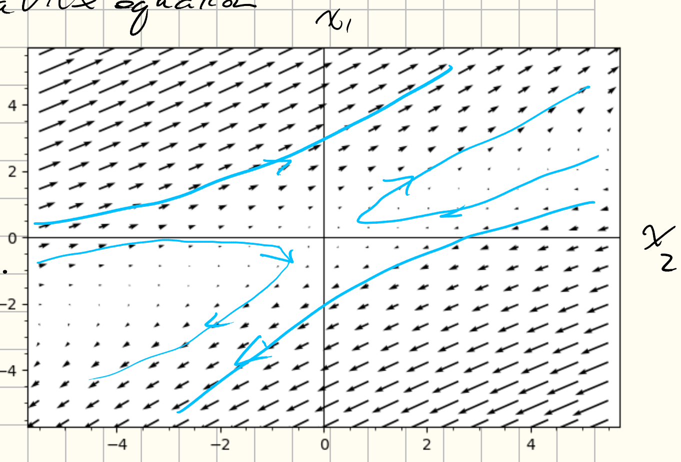
non-linear

$y = x_1$

$x = x_2$

This is linear, we can write it as a matrix equation

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

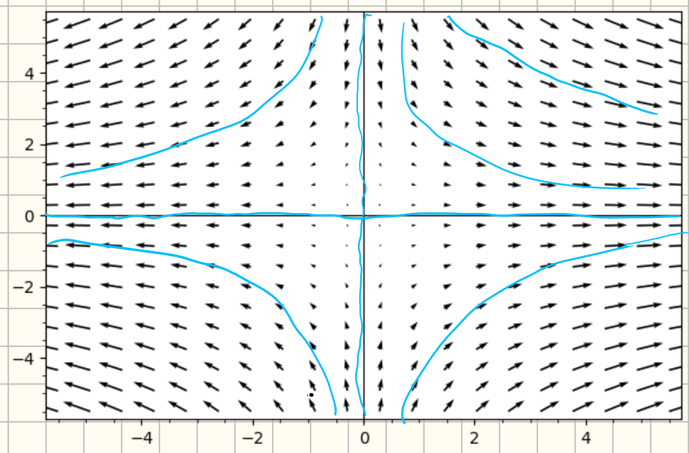


OR $$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \qquad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

We already know these are the wrong coordinates

Consider a different system:

$$\frac{d\mathfrak{z}}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\mathfrak{z}$$



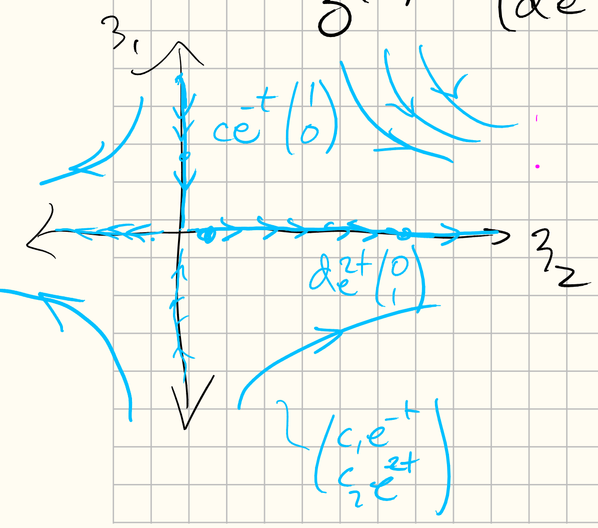
OR
$$\begin{cases} \dfrac{dz_1}{dt} = -z_1 \\ \dfrac{dz_2}{dt} = 2z_2 \end{cases}$$

This system is <u>uncoupled</u> z_1 and z_2 evolve independently

Eg: If we set $z_2 = 0$ this has solutions $z_1 = ce^{-t}$ b/c $\dfrac{dz_1}{dt} = -z_1$

or as a vector $$\mathfrak{z}(t) = \begin{pmatrix} ce^{-t} \\ 0 \end{pmatrix} = ce^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Another solution: $$\mathfrak{z}(t) = \begin{pmatrix} 0 \\ de^{2t} \end{pmatrix} = de^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

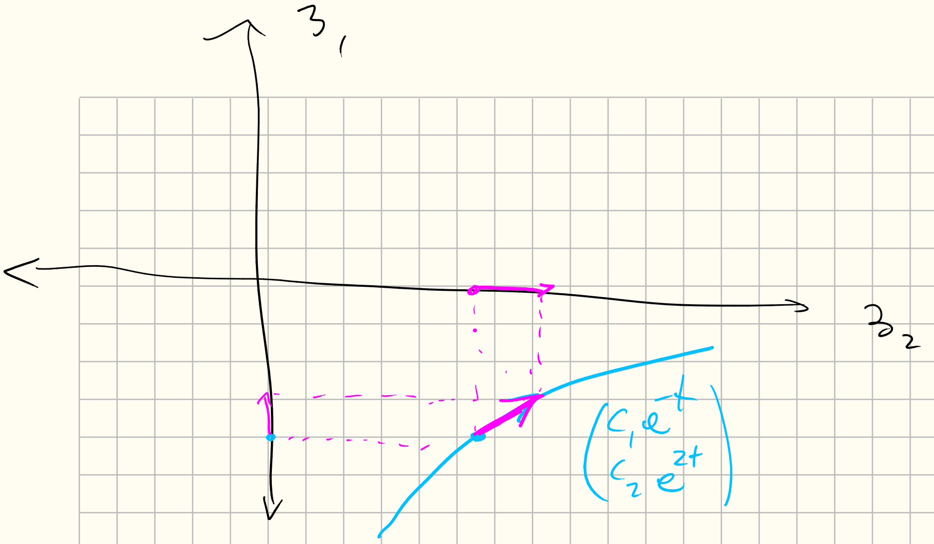


Linearity $\Rightarrow$ Principle of superposition

$\Rightarrow$ We get a solution

$$\mathfrak{z}(t) = c_1e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1e^{-t} \\ c_2e^{2t} \end{pmatrix}$$



We can determine time evolution of each coordinate z_1 & z_2 independently.

Let's go back to

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$$

This doesn't appear to be uncoupled at all!

$$\begin{cases} \dfrac{dx_1}{dt} = 3x_1 - x_2 \\[2ex] \dfrac{dx_2}{dt} = 4x_1 - 2x_2 \end{cases}$$

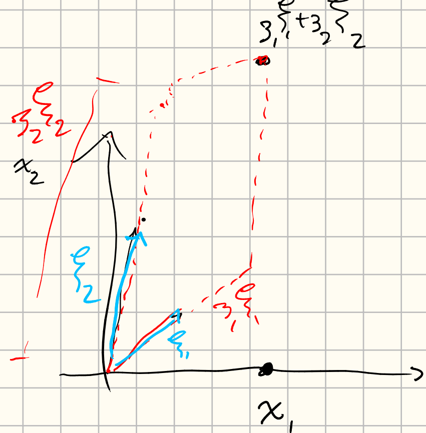
We need both x_1 & x_2 to find the time evolution of $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

These two equations are actually the same, after a linear change of coordinates: In the coordinates corresponding to the eigenvectors of A, the system decouples

Our coordinates z_1 and z_2 are just coefficients of eigenvectors

$\lambda_1 = -1, \quad \lambda_2 = 2$

$\xi_{\lambda_1} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \xi_{\lambda_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



Change of coordinates

$$\left(\xi_{\lambda_1} \middle| \xi_{\lambda_2} \right) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}}_{= P} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

How do we use this?

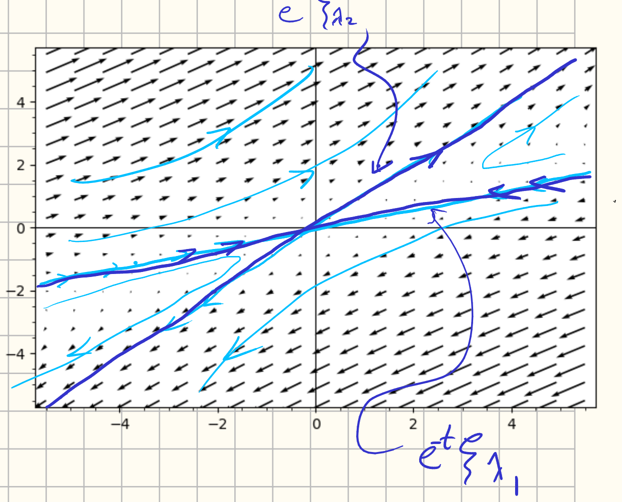
Our solutions are $\mathbf{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{2t} \end{pmatrix}$

But $P\mathbf{z} = \mathbf{x} \Rightarrow$ our solutions $\mathbf{x}(t)$ are

$$P\,\mathbf{z}(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{2t} \end{pmatrix}$$

$$\mathbf{x}(t) = \begin{pmatrix} c_1 e^{-t} + c_2 e^{2t} \\ 4c_1 e^{-t} + c_2 e^{2t} \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$ is the general solution

This is a general method for solving linear systems.



rewrite this:

$$\mathbf{x}(t) = P\,\mathbf{z}(t)$$
$$= \left(\xi_{\lambda_1} \middle| \xi_{\lambda_2} \right) \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$
$$= c_1 e^{\lambda_1 t} \xi_{\lambda_1} + c_2 e^{\lambda_2 t} \xi_{\lambda_2}$$
$$= c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

<u>Theorem</u>: The linear system (homogeneous linear 1st order system of ODEs) (supposing A has distinct eigenvalues)

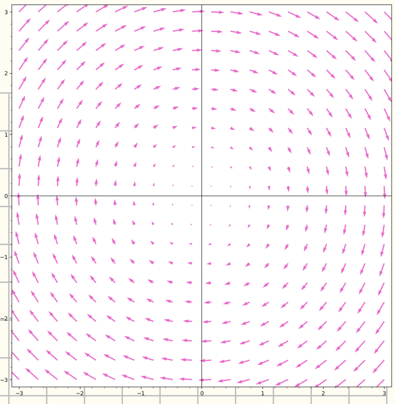
$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

has general solution

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \xi_{\lambda_1} + c_2 e^{\lambda_2 t} \xi_{\lambda_2} + \cdots + c_n e^{\lambda_n t} \xi_{\lambda_n}$$

where $\{\lambda_1, \dots, \lambda_n\}$ are eigenvalues of A with eigenvectors $\{\xi_{\lambda_1}, \dots, \xi_{\lambda_n}\}$.

<u>Rmk</u>: In eigenvector coords., the system decouples.



Eg:
$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Step 1: Eigenvalues

$$\det\begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} = (\cos\theta - \lambda)^2 + \sin\theta$$

$$= \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta \qquad \text{take } \theta = \frac{\pi}{2}$$

$$= 1 + \lambda^2 \qquad \text{roots } i, -i \quad \ldots$$