ODEs 5-17

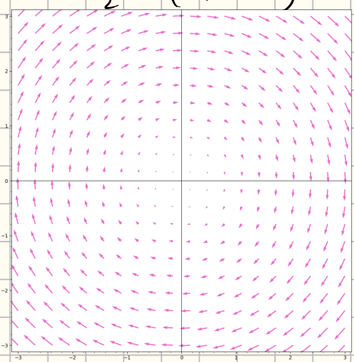
- HW #5 due Monday 11:59 PM ← new HW due time
- Midterm: 48 hours, start 12:30 PM Thursday
- Wednesday: Office hours 5 PM-6 PM (instead of 2-3 pm)

Goal:
- Finish solution of n-dim system (homogeneous, constant coefficients)
- Jordan canonical form, matrix exponential

$y' = ky \rightsquigarrow y = Ae^{kt}$

Works when k is a matrix

Eg: From thursday: $x' = R_\theta x \quad \theta = \frac{-\pi}{2} \quad R_{\frac{\pi}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



$$\begin{cases} \frac{dx_1}{dt} = -x_2 \\ \frac{dx_2}{dt} = x_1 \end{cases}$$

$$\det\left(\begin{pmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{pmatrix}\right) = \lambda^2 + 1 \quad \text{with roots } \pm i.$$

Find complex solution:

$\lambda = +i$:
$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (-i)\mathrm{col}_1 = \mathrm{col}_2$$

$\xi_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ is a $(+i)$-eigenvector

$\lambda = -i$
$$A - \lambda I = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad i\,\mathrm{col}_1 = \mathrm{col}_2$$

$\xi_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ is $(-i)$-eigenvector

General complex solution:

$$x(t) = c_1 e^{\lambda_1 t}\xi_{\lambda_1} + c_2 e^{\lambda_2 t}\xi_{\lambda_2} \qquad c_1, c_2 \in \mathbb{C}$$

$$= c_1 e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{-it}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Strategy: find c_1 & c_2 s.t. the solution is real (enough times to n solutions)
conclude we found real solution using superposition

Recall: $\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$ $\quad \mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$

are linear combinations of z (works for $\mathrm{Re}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \frac{\vec{z}+\overline{\vec{z}}}{2}$)

$$\mathrm{Re}\left(e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix}\right) = \frac{1}{2}e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{1}{2}e^{-it}\begin{pmatrix} -i \\ 1 \end{pmatrix} \text{ is a solution}$$

$$= \mathrm{Re}\begin{pmatrix} ie^{it} \\ e^{it} \end{pmatrix} = \mathrm{Re}\begin{pmatrix} i(\cos(t)+i\sin(t)) \\ \cos(t)+i\sin(t) \end{pmatrix}$$

$$= \mathrm{Re}\left(\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} + i\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}\right)$$

$$= \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$$

$$\mathrm{Im}\left(e^{it}\begin{pmatrix} i \\ 2 \end{pmatrix}\right) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

Conclusion:
The general real solution is $\boxed{x(t) = d_1\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} + d_2\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \quad d_1, d_2 \in \mathbb{R}}$

You hopefully notice a very strong similarity between

2nd order 1-d homogeneous equations

$\updownarrow$

1st order 2-d homogeneous systems of equations

$\boxed{\text{Fundamental principle: Any } n\text{-th order ODE can be rewritten as an equivalent } n\text{-dimensional system of 1st order ODEs}}$

$$\frac{d^n y}{dt^n} = F(t, y, y', \dots, y^{(n-1)})$$

$$\downarrow \qquad z_i = y^{(i)} \text{ for } i = 0, \dots, n-1$$

$$\begin{cases} \frac{dz_0}{dt} = y' = z_1 \\ \frac{dz_1}{dt} = z_2 \\ \vdots \quad = z_{n-1} \\ \frac{dz_{n-1}}{dt} = y^{(n)} = F(t, z_0, z_1, \dots, z_{n-2}) \end{cases}$$

Eg: $y'' + m y' + ky = 0$

$$\downarrow$$

$$\begin{cases} \frac{dy}{dt} = \dot{y} \\ \frac{d\dot{y}}{dt} = -m\dot{y} - ky \end{cases} \rightsquigarrow \begin{pmatrix} y \\ \dot{y} \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -k & -m \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

$y' = \dot{y}$ $\quad \frac{d\dot{y}}{dt} = \ddot{y} = y'' = \dot{y}'$ Our equation is $y'' + my' + ky = 0$

We can determine the highest order derivative from those of lower order:

$$\dot{y}' = -m\dot{y} - ky$$

This is helpful because we've seen that systems can be solved when they decouple:

Eg: Newton's law $F = ma$ decouples in the 3 spatial directions:

$$\vec{F} = m\vec{a} \rightsquigarrow \begin{matrix} x_1'' = \frac{1}{m} F_1 \\ x_2'' = \frac{1}{m} F_2 \\ x_3'' = \frac{1}{m} F_3 \end{matrix} \quad \text{3 2nd order ODEs}$$

We can turn each into a system of 1st order equations

$P_i = m x_i'$ (set $m = 1$)

$$\begin{cases} x_1' = P_1 \\ P_1' = F_1 \end{cases} \quad + \text{ same for } x_2 P_2,\ x_3, P_3.$$

+ Demo about small oscilations in these systems.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'' = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \vec{P} \\ \vec{x} \end{pmatrix} = c_1 \cos(\omega_1 t) \vec{\xi}_1 + c_2 \sin(\omega_1 t) \vec{\xi}_2 + c_3 \cos(\omega_2 t) \vec{\xi}_3 + c_4 \sin(\omega_2 t) \vec{\xi}_4$$

$$\vec{\xi}_i \in \mathbb{R}^4$$

Conclude that A has eigenvalues $-\omega_1^2, -\omega_2^2$ b/c

forget $\vec{P}$ $\to$

$$(\vec{x})'' = -\omega_1^2 c_1 \cos(\omega_1 t) \begin{pmatrix} \vdots \\ \vec{\xi}_{1,x} \end{pmatrix} + \dots$$

$$= A\vec{x}$$

$\Rightarrow$ 2 uncoupled oscilators w/ frequency ω_1 & ω_2 : $\begin{matrix} x_1'' = -\omega_1^2 x_1 \\ x_2'' = -\omega_2^2 x_2 \end{matrix}$

after a linear change of coordinates

Let's go back to solving systems of 1st order equations

Eg: Use the techniques to solve

$$y'' + y' + \frac{5}{4}y = 0$$

$$\vec{x} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} \qquad \vec{x}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{5}{4} & -1 \end{pmatrix}}_{A} \left(\vec{x}\right)'$$

$$\dot{y}' = -\dot{y} - \frac{5}{4}y$$

characteristic polynomial : $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{5}{4} & -\lambda - 1 \end{vmatrix}$

$$= \lambda^2 + \lambda + \frac{5}{4}$$

roots $\frac{-1}{2} + i$, $\frac{-1}{2} - i$

$\lambda_1 = -\frac{1}{2} + i$

$$A - \lambda_1 I = \begin{pmatrix} \frac{1}{2} - i & 1 \\ -\frac{5}{4} & -\frac{1}{2} - i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\begin{pmatrix} \frac{1}{2} - i & 1 \\ -\frac{5}{4} & -\frac{1}{2} - i \end{pmatrix} \xrightarrow{r_1 \mapsto \left(\frac{1}{2} + i\right) r_1} \begin{pmatrix} \frac{5}{4} & \frac{1}{2} + i \\ -\frac{5}{4} & -\frac{1}{2} - i \end{pmatrix} \xrightarrow{r_2 \mapsto r_2 + r_1} \begin{pmatrix} \frac{5}{4} & \frac{1}{2} + i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\left(\frac{1}{2} - i\right)\left(\frac{1}{2} + i\right) \quad \frac{1}{4} + 1 = \frac{5}{4}$$

$$a = \frac{1}{2} + i \qquad b = -\frac{5}{4}$$

$$\xi_1 = \begin{pmatrix} \frac{1}{2} + i \\ -\frac{5}{4} \end{pmatrix}$$

Complex solution: $x = c_1 e^{\left(-\frac{1}{2} + i\right)t} \xi_1 + c_2 e^{\left(-\frac{1}{2} - i\right)t} \xi_2 \qquad \xi_2 = \overline{\xi_1}$

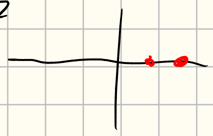
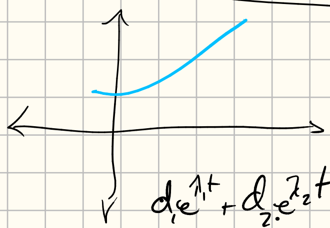
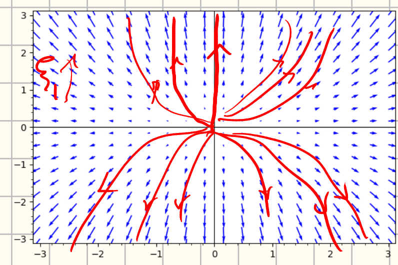
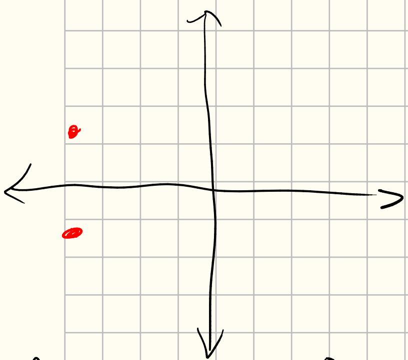
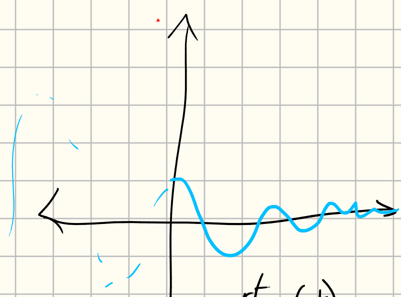
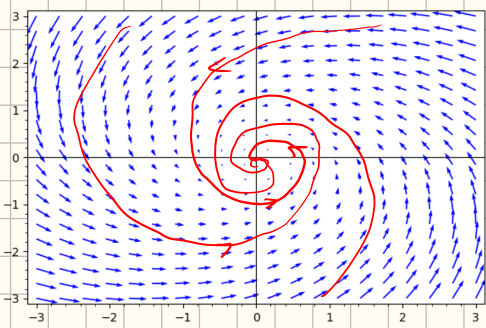
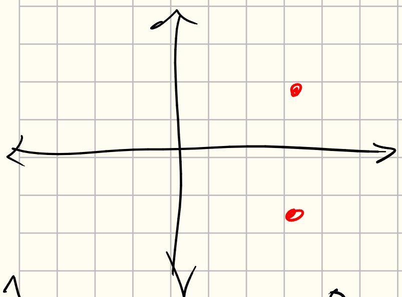
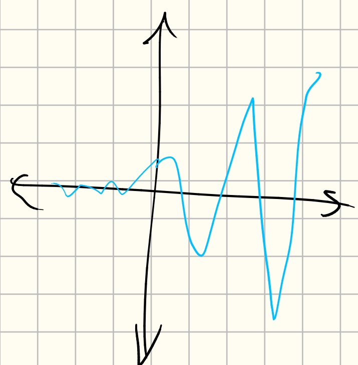
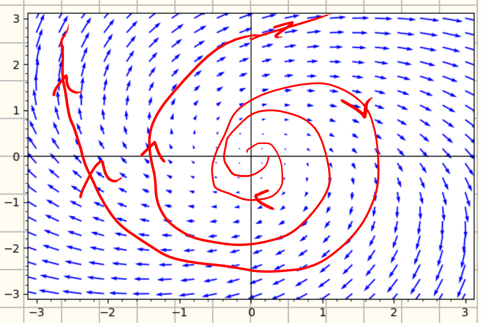
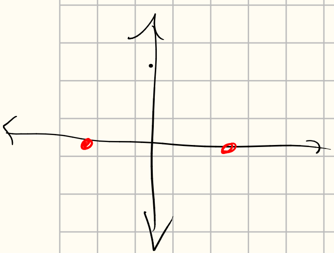
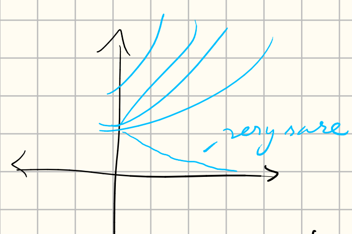
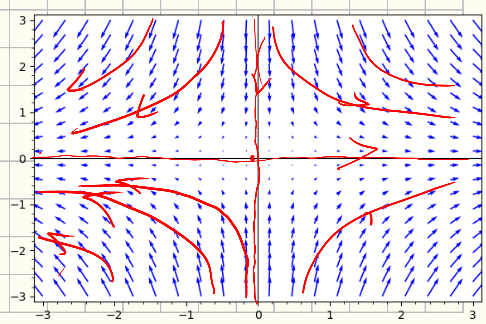
$c_1, c_2 \in \mathbb{C}$

Real solution: $x(t) = d_1 \operatorname{Re}\left(e^{\left(-\frac{1}{2} + i\right)t} \xi_1\right) + d_2 \operatorname{Im}\left(e^{\left(-\frac{1}{2} + i\right)t} \xi_2\right)$

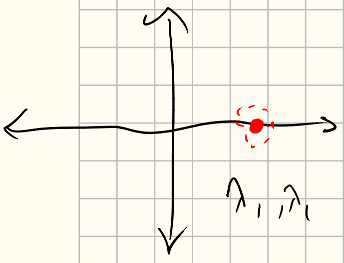
$$x(t) = d_1 e^{-\frac{1}{2}t} \cos(t) \begin{pmatrix} \frac{1}{2} \\ -\frac{5}{4} \end{pmatrix} + d_2 e^{-\frac{1}{2}t} \sin(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; d_1, d_2 \in \mathbb{R}.$$

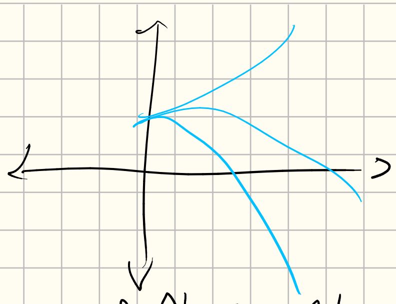
The study of n-dim systems (homogeneous, linear, constant coefficients)
is almost identical to the study of nth order homogeneous equations

Only difference comes from repeated roots.

Roots of characteristic polynomial	2nd order ODE	2-d system of 1st order
$\lambda_1 > \lambda_2 > 0$ 	 $d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t}$	 $d_1 e^{\lambda_1 t}\xi_1 + d_2 e^{\lambda_2 t}\xi_2$ Asymptotically all solutions $\sim e^{\lambda_1 t}\xi_1$
 $\lambda_1 = -r + i\omega$ $r > 0$ $\lambda_2 = \overline{\lambda_1}$	 $d_1 e^{-rt}\cos(\omega t) + d_2 e^{-rt}\sin(\omega t)$	 $d_1 e^{-rt}\cos(\omega t)\xi_1 + d_2 e^{-rt}\sin(\omega t)\xi_2$
 $\lambda_1 = +r + i\omega$ $r > 0$ $\lambda_2 = \overline{\lambda_1}$		 $d_1 e^{rt}\cos(\omega t)\xi_1 + d_2 e^{rt}\sin(\omega t)$
 $\lambda_1 < 0 < \lambda_2$	 $d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t}$	 $d_1 e^{\lambda_1 t}\xi_1 + d_2 e^{\lambda_2 t}\xi_2$

What about repeated roots? Here behavior diverges slightly

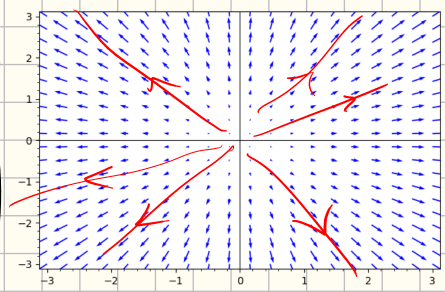




$d_1 e^{\lambda t} + d_2 t e^{\lambda t}$

don't correspond

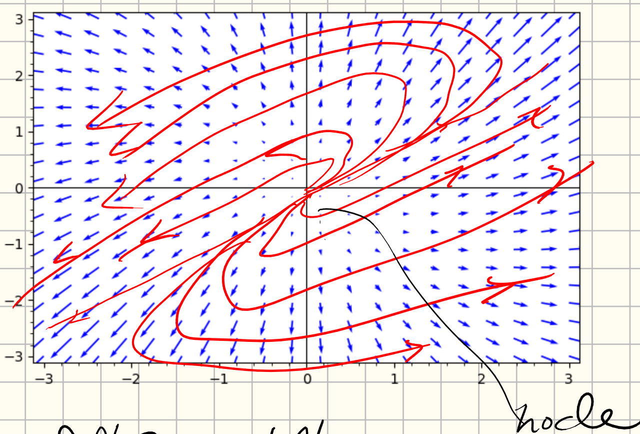
→ 2 cases



$d_1 e^{\lambda t}\xi_1 + d_2 e^{\lambda t}\xi_2$

$\xi_1 \neq \xi_2$ both eigenvectors

Case that corresponds here



Solution: $d_1 e^{\lambda t}\xi_1 + d_2 t e^{\lambda t}\eta_1$

Where η_1 is a generalized eigenvector.

To cover this case, a few definitions:

Given a matrix A

characteristic polynomial $p(\lambda) = \det(A - \lambda I)$

Def: The algebraic multiplicity of λ is the order of the root of λ in $p(\lambda)$

Eg: $p(\lambda) = (\lambda - 2)^3$, algebraic multiplicity of 2 is 3

Def: The geometric multiplicity is the # of eigenvectors w/ that eigenvalue, sometimes $<$ algebraic multiplicity

always geometric $\leq$ algebraic

Eg: $A = \begin{pmatrix} 3&0&0&0 \\ 0&2&1&0 \\ 0&0&2&1 \\ 0&0&0&2 \end{pmatrix}$

$$\det(A-\lambda I) = (\lambda-3)(\lambda-2)^3$$

There is only $\underline{\underline{1}}$ $\lambda = 2$ eigenvector:

$$(A-2I)\xi = 0$$

$\begin{pmatrix} 3&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}\xi = 0$ only has solution $\xi = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

But $(A-2I)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \xi$ $\Rightarrow$ $(A-2I)^2\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = (A-2I)\xi = 0$

~~B/c~~ $\begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Def: A vector η s.t. $(A-\lambda I)^k \eta = 0$ is a <u>generalized eigenvector</u>

Thm: If we have an equation $x' = Ax$

and A only has one $\lambda = \lambda_1$ eigenvector ξ_1

w/ corresponding solution $e^{\lambda_1 t}\xi_1$

then if η_1 is a solution to $(A-\lambda_1 I)\eta_1 = \xi_1$ ← use G.E.

then $e^{\lambda_1 t}\eta_1 + te^{\lambda_1 t}\xi_1$ is also a solution.