ODE 5-18

Goal:
- Finish $\vec{x}' = A\vec{x}$ (recipe for solution, 2 recipes)
- Inhomogeneous problem $\vec{x}'(t) = A(t)\vec{x}(t) + \vec{g}(t)$
- Learn more linear algebra

Principle: If geometric multiplicity < algebraic multiplicity then the system doesn't quite decouple, coupling is controllable. Best we can do: eigenvectors + generalized e.vects.

Eg: $x' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} x$ $\qquad A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$

char poly $\det\begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) + 1$

$= \lambda^2 - 4\lambda + 4 = (\lambda-2)^2.$

$\lambda = 2$: $A - 2I = \begin{pmatrix} 1-2 & -1 \\ 1 & 3-2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ so

$\vec{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a 2-e.vector. $\qquad \xi$ is the only one

$c_1 e^{\lambda t}\vec{\xi}$ is a solution

Generalized eigenvector: Solve $(A - 2I)\vec{\eta} = \vec{\xi}$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\qquad \left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right) \xrightarrow{r_2 \to r_2 + r_1} \left(\begin{array}{cc|c} -1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right).$

$\eta_2 = 0$, $-\eta_1 = 1$ $\qquad \eta_1 = -1$

$\vec{\eta} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ satisfies $(A - 2I)\vec{\eta} = \vec{\xi} \Rightarrow (A-2I)^2\vec{\eta} = 0$

$\left(\begin{array}{ccc|c} a & b & \cdots & y_1 \\ & c & & \vdots \\ 0 & & \ddots & y_n \end{array}\right)$. $\qquad$ $\vdots$ $a_{n-1\,n-1}x_{n-1} + a_{n-1\,n}x_n = y_{n-1}$, $a_{nn}x_n = y_n$ $\qquad$ find x_n using last row, then solve x_{n-1}

Solution: $\boxed{c_1 e^{\lambda t}\vec{\xi}_1 + c_2\left(te^{\lambda t}\vec{\xi}_1 + e^{\lambda t}\vec{\eta}_1\right)}$

$\vec{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left(te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)$

We're done with $\vec{x}' = A\vec{x}$

Recipe:

Step 1: find roots $\{\lambda_1, \dots, \lambda_n\}$ of $\det(A - \lambda I)$

Step 2: Find eigenvectors $\{\vec{\xi}_{\lambda_i}\}$

Solutions $c_1 e^{\lambda_1 t}\vec{\xi}_{\lambda_1} + c_2 e^{\lambda_2 t}\vec{\xi}_{\lambda_2} + \dots$ | take $\mathrm{Re}(-)$, $\mathrm{Im}(-)$ to find real solutions

Step 3 if # e.vectors < # e.values

find $\vec{\eta}_{\lambda_i}$ solving $(A - \lambda_i I)\vec{\eta}_{\lambda_i} = \vec{\xi}_{\lambda_i}$

– Add solutions

$$\tilde{c}_i \left(t e^{\lambda_i t}\vec{\xi}_i + e^{\lambda_i t}\vec{\eta}_i \right)$$

Step 4 (if we're not done) find

$\vec{\eta}_i^{(2)}$ s.t. $(A - \lambda_i I)\vec{\eta}_i^{(2)} = \vec{\eta}_i$ $\qquad (A - \lambda_i I)^3 \vec{\eta}_i^{(2)} = 0$

– Add solution

$$\tilde{\tilde{C}}_i \left(\frac{t^2}{2} e^{\lambda_i t}\vec{\xi}_i + t e^{\lambda_i t}\vec{\eta}_i + e^{\lambda_i t}\eta_i^{(2)} \right)$$

$\vdots$

$$\tilde{\tilde{C}}_i \left(\frac{t^n}{n!} e^{\lambda_i t}\vec{\xi}_i + \dots + \frac{t^k}{k!} e^{\lambda_i t}\eta_i^{(\cdot)} + \dots + e^{\lambda_i t}\eta_i^{(n-1)} \right)$$

Eg: $\vec{x}' = A\vec{x}$ really doesn't decouple. $\left(\begin{array}{l} \text{alg. mult} = 3 \\ \text{geo. mult.} = 1 \end{array} \right)$

$$A = \begin{pmatrix} \frac{3}{2} & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 3/2 \end{pmatrix}$$

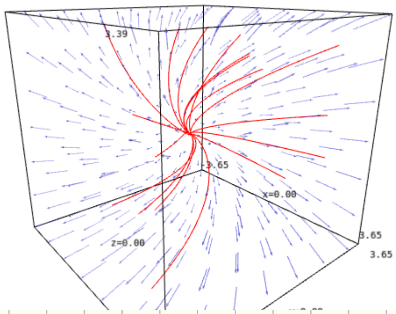
$\vec{\xi} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an $\frac{3}{2}$-eigenvector

$\vec{\eta} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and

$\vec{\eta}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are generalized e.vects

$A(\vec{\eta}^{(2)}) = \frac{3}{2}\vec{\eta}^{(2)} + \vec{\eta}$

This is a node: the system doesn't decouple.



regardless: our solution

$$c_1 e^{\frac{3}{2}t}\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\left(te^{\frac{3}{2}t}\begin{pmatrix}1\\0\\0\end{pmatrix} + e^{\frac{3}{2}t}\begin{pmatrix}0\\1\\0\end{pmatrix}\right) + c_3\left(\frac{t^2}{2}e^{\frac{3}{2}t}\begin{pmatrix}1\\0\\0\end{pmatrix} + te^{\frac{3}{2}t}\begin{pmatrix}0\\1\\0\end{pmatrix} + e^{\frac{3}{2}t}\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

Fundamental matrices and IVPs

Let $\vec{x}\,' = A(t)\vec{x}$ - linear homogenous 1st order system

Def: A fundamental matrix (for this system) is a matrix $\Phi(t)$ s.t. $\vec{x}\,' = A(t)\vec{x}$, $\vec{x}(t_0) = \vec{x}_0$ has solution $\Phi(t)\vec{x}_0$.

These are easy to write down: let $\{\vec{x}_1(t), \ldots, \vec{x}_n(t)\}$ be a fundamental set of solutions. Let $\Psi(t) = \left(\vec{x}_1 \middle| \vec{x}_2 \middle| \cdots \middle| \vec{x}_n(t)\right)$ so our general solution is

$$\Psi(t)\vec{c} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n.$$

Relate $\vec{c}$ to $\vec{x}_0$ (in practice: use G.E.)

$\vec{x}(t_0) = \Psi(t_0)\vec{c}$ so if $\vec{c} = \Psi(t_0)^{-1}\vec{x}_0$ then

$\Psi(t)\vec{c} = \Psi(t)\Psi(t_0)^{-1}\vec{x}_0$ solves the IVP

Conclusion $\boxed{\Phi(t) = \Psi(t)\Psi(t_0)^{-1}}$ is a fundamental matrix.

Check that a set of solutions is fundamental. The only thing that could go wrong is $\Psi(t_0)^{-1}$ might not exist. It does exist if

$$\det(\Psi(t_0)) \neq 0.$$

Def: The Wronskian $W(t_0)$ is

$$W[\vec{x}_1, \ldots \vec{x}_n](t_0) = \det\left(\vec{x}_1 \middle| \cdots \middle| \vec{x}_n\right).$$

Conclusion: $W(t_0) \neq 0 \iff \{\vec{x}_1, \dots, \vec{x}_n\}$ is a fundamental set of sol'ns.

Rmk: If we nth order 1d ODE $\rightsquigarrow$ 1st order n-d system
definitions of $W[x_1, \dots, x_n](t_0)$ are equivalent

Inhomogeneous systems of equations

$$\vec{x}' = A(t)\vec{x} + \vec{g}(t)$$

Principles are the same as in the 1d case.

Linearity $\Rightarrow$ superposition $\Rightarrow$

If $\vec{x}_p(t)$ is any solution to $\vec{x}' = A(t)\vec{x} + g(t)$ and

$\Psi(t)\vec{c}$ is the general solution to the complementary homogeneous equation

$$\vec{x}' = A(t)\vec{x}$$

Then the general solution is

$$\boxed{\begin{aligned}\vec{x}(t) &= \Psi(t)\vec{c} + \vec{x}_p(t) \\ &= \underbrace{c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \dots + c_n\vec{x}_n(t)}_{\text{complementary solution}} + \underbrace{\vec{x}_p(t)}_{\text{Particular solution}}\end{aligned}}$$

How do we find $\vec{x}_p(t)$?

Several Methods

Eg: 1 Guessing (Undetermined coefficients)

Works if $\vec{g}(t)$ is $\left\{ \begin{array}{l} t^k \vec{y}_1 \\ \cos(\omega t), \sin(\omega t)\vec{y}_3 \\ e^{\alpha t}\vec{y}_2 \end{array} \right.$

Ansatz $\vec{x}_p(t) = At^k\vec{y}_1 + B\cos(\omega t)\vec{y}_3$, etc...

Eg2: Method of variation of parameters

Let $\Psi(t)\vec{c}$ be the general solution to $\vec{x}' = A(t)\vec{x}$

Ansatz $\vec{x}_p(t) = \Psi(t)\vec{u}(t) = \Psi(t)\begin{pmatrix} \vec{u}_1(t) \\ \vec{u}_2(t) \\ \vdots \\ \vec{u}_n(t) \end{pmatrix}$

Claim: This always works.

Proof: $\vec{x}' = A(t)\vec{x} + g(t)$

$$[\Psi(t)u(t)]' = A(t)\Psi(t)u(t) + g(t)$$

$$\Psi'(t)\vec{u}(t) + \Psi(t)\vec{u}(t)' = A(t)\Psi(t)u(t) + g(t) \qquad \text{since } \Psi'(t) = A(t)\Psi(t)$$

$$(x_1'|x_2'|\dots) = A(t)(x_1|\dots|x_n)$$

$$\Psi(t)\vec{u}(t)' = g(t)$$

$$\vec{u}(t)' = \Psi(t)^{-1}g(t)$$

$$\vec{u}(t) = \int_{t_0}^{t} \Psi(s)^{-1}g(s)\,ds$$

$$\boxed{\vec{x}_p = \Psi(t)\vec{u}(t) = \Psi(t)\int_{t_0}^{t}\Psi(s)^{-1}g(s)\,ds}$$

variation of parameters.

Rmk: In practice, usually calculate $\underbrace{\Psi(s)^{-1}g(s)}_{\gamma(s)}$ using G.E. to solve

$$\Psi(s)\gamma(s) = g(s).$$

Eg: Solve $\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$

Step 1: Complementary solution $c_1 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\therefore \Psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$$

Particular solution is $\Psi(t)\int_{t_0}^{t}\Psi(s)^{-1}g(s)\,ds = \vec{x}_p(t)$

Solve for $\Psi(s)^{-1}g(s)$ $\left(\begin{array}{cc|c} e^{-3t} & e^{-t} & 2e^{-t} \\ -e^{-3t} & e^{-t} & 3t \end{array}\right) \xrightarrow{r_2 \to r_2 + r_1} \left(\begin{array}{cc|c} e^{-3t} & e^{-t} & 2e^{-t} \\ 0 & 2e^{-t} & 3t + 2e^{-t} \end{array}\right)$

$$\gamma(s) = \begin{pmatrix} \gamma_1(s) \\ \gamma_2(s) \end{pmatrix} \qquad \Psi\vec{\gamma} = \vec{g}$$

$$2e^{-t}\gamma_2(s) = 3t + 2e^{-t}$$

$$\gamma_2(s) = \frac{3}{2}te^{t} + 1$$

$$e^{-3t}\gamma_1(t) + e^{-t}\gamma_2(t) = 2e^{-t}$$

$$e^{-3t}\gamma_1(t) + e^{-t}\left(\frac{3}{2}te^{t} + 1\right) = 2e^{-t}$$

$$\gamma_1(t) = e^{3t}\left(2e^{-t} - e^{-t}\left(\frac{3}{2}te^{t} + 1\right)\right) = e^{2t} - \frac{3}{2}te^{3t}$$

$$\gamma(s) = \begin{pmatrix} e^{2s} - \frac{3}{2} s e^{3s} \\ \frac{3}{2} s e^{s} + 1 \end{pmatrix}$$

$$\vec{x}_p = \Psi(t) \int_{t_0}^{t} \begin{pmatrix} e^{2s} - \frac{3}{2} s e^{3s} \\ \frac{3}{2} s e^{s} + 1 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} \int_0^t e^{2s} - \frac{3}{2} s e^{3s} \, ds \\ \int_0^t \frac{3}{2} s e^{s} + 1 \, ds \end{pmatrix}$$

Fundamental matrices for constant coefficient systems

(Matrix exponential & Jordan canonical form)

Goal: find $\Phi(t)$ for $\vec{x}' = A\vec{x}$

Easiest in decoupled case $\vec{x}' = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \vec{x}$

then $\Phi(t) = \begin{pmatrix} e^{a_1 t} & & & 0 \\ & e^{a_2 t} & & \\ & & \ddots & \\ 0 & & & e^{a_n t} \end{pmatrix}$ is a fundamental matrix.

In general:

$$\boxed{\Phi(t) = e^{tA} \text{ is a fundamental matrix for } x' = Ax}$$

Where

$$e^{tA} = \sum_{n=0}^{\infty} \frac{1}{n!} (tA)^n$$

matrix multiplication

Because 1: $\Phi'(t) = A\Phi(t)$ since

$$\Phi'(t) = \left(\sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n \right)' = \sum_{n=0}^{\infty} \frac{n}{n!} t^{n-1} A^n = A \left(\sum_{n'=0}^{\infty} \frac{1}{n'!} t^{n'} A^{n'} \right) = A\Phi(t)$$

2: $\Phi(0) x_0 = \vec{x}(0)$

$$\Phi(0) = I + tA + \frac{t^2}{2} A^2 + \dots \Big|_{t=0} = I$$

so $\Phi(0) x_0 = \vec{x}(0)$.

Cave emptor:

- $e^A e^B \neq e^B e^A \neq e^{B+A}$ if $AB \neq AB$
- $\frac{d}{dt} e^{A(t)} \neq \frac{d}{dt} A(t) \, e^{A(t)}$.

Important facts:

- e^{tA} is easy for diagonal matrices (i.e. uncoupled systems)

$$e^{t\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}} = \begin{pmatrix} e^{ta_1} & 0 \\ 0 & e^{ta_2} \end{pmatrix} \quad \text{b/c} \quad \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}^n = \begin{pmatrix} a_1^n & 0 \\ 0 & a_2^n \end{pmatrix}$$

$$= \begin{pmatrix} \sum \frac{t^n}{n!} a_1^n & 0 \\ 0 & \sum \frac{t^n}{n!} a_2^n \end{pmatrix}$$

- Exponential is easy on block diagonal matrices $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ A, B, C, D are also matrices

$$e^{t\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)} = \left(\begin{array}{c|c} e^{tA} & 0 \\ \hline 0 & e^{tB} \end{array}\right)$$

hopefully clear since different blocks decouple from eachother

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)^n = \left(\begin{array}{c|c} A^n & 0 \\ \hline 0 & B^n \end{array}\right)$$

- On a Jordan block $J_\lambda = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$ the exponential is

$$J_\lambda = \underbrace{\begin{pmatrix} \lambda & 0 & 0 & 0 \\ & \lambda & 0 & 0 \\ & & \lambda & 0 \\ 0 & & & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{AB = BA} \quad \text{and} \quad e^{t\begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} + t\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & 0 & 1 \\ 0 & & & 0 \end{pmatrix}} = e^{t\lambda I} e^{t\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

$$e^{tJ_\lambda} = \begin{pmatrix} e^{t\lambda} & & 0 \\ & e^{t\lambda} & \\ 0 & & \ddots & \\ & & & e^{t\lambda} \end{pmatrix} \left[I + \begin{pmatrix} 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ 0 & & 0 & 0 \\ & & & 0 \end{pmatrix} + \frac{t^3}{6} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right]$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ 0 & & 0 & 0 \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

General A (Next time)

If A is block diagonal Jordan blocks

$$A = \left(\begin{array}{c|c|c} J_1 & 0 & 0 \\ \hline 0 & J_2 & 0 \\ \hline 0 & 0 & J_3 \end{array}\right)$$

then we have a nice formula

Eg: S.l to $\vec{x}\,' = \begin{pmatrix} 3/2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 3/2 \end{pmatrix} \vec{x}$ is

$$e^{t\begin{pmatrix} 3/2 & 1 & 0 \\ & 3/2 & 1 \\ 0 & & 3/2 \end{pmatrix}} \vec{x}(0) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2} e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix} \vec{x}(0)$$