ODE 5-19

Goal: • Finish studying matrix exponential
• Start to learn about non-linear systems & stability

Recall

• $\vec{X}' = A\vec{x}$ has solution $\vec{x}(t) = e^{tA}\vec{x}(0)$

• We can compute e^{tJ_λ} for a Jordan block J_λ.

$$J_\lambda = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Analyze matrices of the form

$$J_\lambda - \lambda I = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & & 0 \end{pmatrix}. \text{ How does this act on } \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$J_\lambda - \lambda I \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_2 \\ \vdots \\ x_{n-1} \\ x_n \\ 0 \end{pmatrix}$$

$$(J_\lambda - \lambda I)^k \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_k \\ \vdots \\ x_n \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad J_\lambda - \lambda I^k = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 0 & \\ & 0 & & 0 & 1 & \\ & & & & 0 & \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is the only λ-eigenvector

J_λ^n is not extremely easy to calculate

$$e^{tJ_\lambda} = \sum \frac{t^n}{n!} J_\lambda^n \text{ has a nice formula.} \qquad \lambda I \cdot (J_\lambda - \lambda I) = (J_\lambda - \lambda I)\lambda I$$

$$e^{tJ_\lambda} = e^{t\begin{pmatrix} \lambda & & 0 \\ 0 & & \lambda \end{pmatrix}} \cdot e^{t\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & 0 & 0 & 1 \\ & & & 0 \end{pmatrix}}$$

$$= \begin{pmatrix} e^{t\lambda} & & 0 \\ 0 & & e^{t\lambda} \end{pmatrix}\left[I + t\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ 0 & 0 & 0 & 1 \\ & & & 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ 0 & & 0 & 0 \end{pmatrix} + \frac{t^3}{6}(\;) + \cdots + \frac{t^n}{n!}\begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{pmatrix}\right]$$

Def
A matrix of the form

$$\begin{pmatrix} J_{\lambda_1} & 0 & 0 & 0 \\ 0 & J_{\lambda_2} & 0 & 0 \\ 0 & 0 & J_{\lambda_3} & 0 \\ 0 & 0 & 0 & J_{\lambda_4} \end{pmatrix}$$

is said to be in Jordan canonical form.

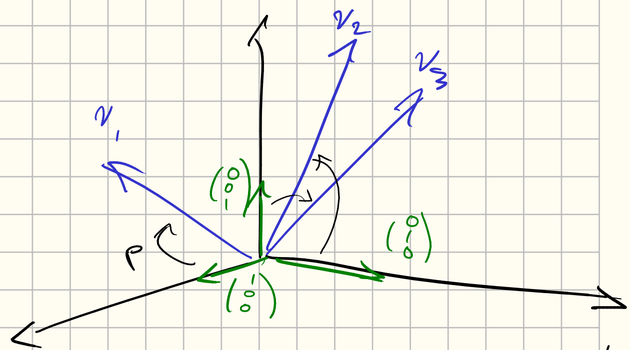
Eg: $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{pmatrix}$ is in Jordan canonical form.

We want to replace A with one in Jordan canonical form via a linear change of coordinates (ODE Perspective: decouple our system $x' = Ax$ as much as possible)

Def: The change of basis matrix for a basis (independent spanning set) $v_1, v_2, \dots, v_n$ is a matrix P so that

$$P\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = v_1 \qquad P\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = v_2, \quad \dots$$

i.e. $P = \left(v_1 \middle| v_2 \middle| \cdots \middle| v_n \right)$



Proposition: If L is a linear map which in coordinates v is given by the matrix

$$A = \begin{array}{c|c} & \begin{matrix} v_1 & \cdots & v_n \end{matrix} \\ \hline \begin{matrix} v_1 \\ \vdots \\ v_n \end{matrix} & \begin{pmatrix} a_{11} & & \\ a_{21} & & \cdots \\ \vdots & & \\ a_{n1} & & \end{pmatrix} \end{array}$$

$A v_1 = \sum a_{i1} v_i$ etc.

In standard coordinates L is given by the matrix

$$PAP^{-1}$$

PS: We want a matrix that sends v_1 to $\sum a_{i1} v_i$ etc.

$$PAP^{-1} v_1 = PA \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{since} \quad P\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = v_1$$

$$= P\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$$

$$= a_{11} v_1 + a_{21} v_2 + \cdots + a_{n1} v_n$$

+ similarly for the other columns.

Thm: For any matrix A there is a matrix P such that PAP^{-1} is in Jordan canonical form.

Pf: Pick a basis of eigenvectors and generalized eigenvectors. (Rmk P may be Complex) Pick P^{-1} to be the change of basis matrix. Because our original matrix is in standard coordinates

$$\text{standard} \underset{P}{\overset{P^{-1}}{\rightleftarrows}} \text{Basis of eigenvectors}$$

Conclusion

$P^{-1}AP$ is in Jordan canonical form if $P = \left(\vec{\xi}_1 \middle| \vec{\eta}_1^{(1)} \middle| \cdots \middle| \vec{\xi}_2 \middle| \vec{\xi}_3 \middle| \vec{\eta}_1^{(1)} \middle| \vec{\eta}_2^{(2)}\right)$

Check: $P^{-1}AP \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} = P^{-1}A\vec{\xi}_i = P^{-1}\lambda_i\vec{\xi}_i = \lambda_i\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$

The system $\vec{x}' = P^{-1}AP\vec{x}$ is as decoupled as possible: $\vec{z}(t) = P\vec{x}(t)$ then

$\vec{z}' = A\vec{z}$ is our original system

This is useful because we can now solve any

$$\vec{x}' = A\vec{x} \quad \text{with matrix multiplication}$$

Let $J = P^{-1}AP$ so $A = PJP^{-1}$ then

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!}(PJP^{-1})^n \qquad \text{since } (PJP^{-1})^n = PJP^{-1}PJP^{-1}\cdots PJP^{-1} = PJ^nP^{-1}$$

$$= P\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}J^n\right)P^{-1}$$

$= Pe^{tJ}P^{-1}$ we get $\vec{x} = e^{tA}\vec{x}_0$ or

$$\boxed{\vec{x} = Pe^{tJ}P^{-1}\vec{x}_0}$$

Eg: Solve $\vec{x}' = \begin{pmatrix}3 & -4\\1 & -1\end{pmatrix}\vec{x}$

$\det(A-\lambda I) = (3-\lambda)(-1-\lambda)+4 = \lambda^2-2\lambda+1 = (\lambda-1)^2$

$\lambda=1$
$A-I$ $\begin{pmatrix}2 & -4\\1 & -2\end{pmatrix}$ $\quad \vec{\xi} = \begin{pmatrix}2\\1\end{pmatrix}$ is a 1-eigenvector

$(A-\lambda I)\vec{\eta} = \vec{\xi} \qquad \left(\begin{array}{cc|c}2 & -4 & 2\\1 & -2 & 1\end{array}\right) \to \left(\begin{array}{cc|c}1 & -2 & 1\\1 & -2 & 1\end{array}\right) \to \left(\begin{array}{cc|c}1 & -2 & 1\\0 & 0 & 0\end{array}\right) \quad \vec{\eta} = \begin{pmatrix}1\\0\end{pmatrix}$

$$P = \left(\vec{\xi} \,\middle|\, \vec{\eta}\right) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$J = P^{-1}AP$ we could matrix multiply. Better: J is a Jordan block

$$J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{because} \quad P^{-1}AP\begin{pmatrix}1\\0\end{pmatrix} = 1\begin{pmatrix}1\\0\end{pmatrix}$$

$$P^{-1}AP\begin{pmatrix}0\\1\end{pmatrix} = P^{-1}A\vec{\eta} = P^{-1}(\vec{\eta}+\vec{\xi}) = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc} \vec{\xi} & \vec{\eta} \end{array} \\ \begin{array}{c} \vec{\xi} \\ \vec{\eta} \end{array} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} \xi & \eta^{(1)} & \eta^{(2)} & \xi_2 \end{array} \\ \begin{array}{c} \xi \\ \eta' \\ \eta^2 \\ \\ \end{array} & \left(\begin{array}{ccc|c} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ \hline 0 & 0 & 0 & \lambda_2 \end{array}\right) \end{array}$$

We know our answer: $\vec{x} = e^{tA}\vec{x}_0 = Pe^{tJ}P^{-1}\vec{x}_0$

$$= Pe^{t\lambda}e^{t(J_\lambda - \lambda I)}P^{-1}\vec{x}_0$$

$$= Pe^t\left(I + t\begin{pmatrix}0&1\\0&0\end{pmatrix} + 0\right)P^{-1}\vec{x}_0$$

$$= P\begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}P^{-1}\vec{x}_0 = \begin{pmatrix}2&1\\1&0\end{pmatrix}\begin{pmatrix} e^t & te^t \\ 0 & e^t\end{pmatrix}\underbrace{\begin{pmatrix}0&-1\\-1&2\end{pmatrix}}_{\det P}\vec{x}_0$$

$$\boxed{\vec{x} = \begin{pmatrix}2&1\\1&0\end{pmatrix}\begin{pmatrix} e^t & te^t \\ 0 & e^t\end{pmatrix}\begin{pmatrix}0&-1\\-1&2\end{pmatrix}\vec{x}_0}$$

Eg: Use matrix exponential to solve

$$y'' - 4y' + 4y = 0$$

$$r^2 - 4r + 4 = 0 \qquad (r-2)^2 = 0 \qquad r = 2, 2$$

If we weren't clever enough to guess te^{rt} then we could have solved this equation this way.

Rewrite as a system of 1st order equations

$$x_1 = y \qquad x_2 = y'$$

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = 4x_2 - 4x_1 \end{cases} \quad \text{or} \quad \vec{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \vec{x}.$$

If we find P

$$\vec{X} = P e^{tJ} P^{-1} \vec{X}_0$$

$$\vec{x} = P \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} P^{-1} \vec{x}_0$$

$$\begin{pmatrix} y \\ y' \end{pmatrix} = P \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} P^{-1} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

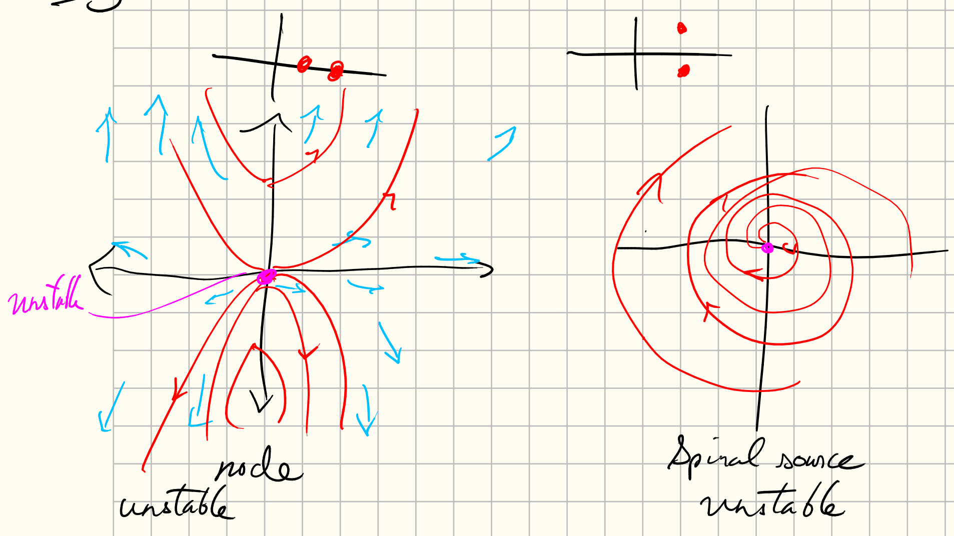
We rewrote our equation as a system & decoupled as much as possible.

Now: Apply the theory of linear systems to non-linear problems

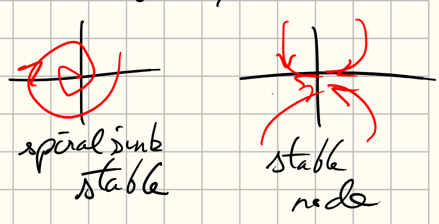
Principle: for autonomous systems, the behavior of a system of ODEs behaves like a linear system nearby a equilibrium point.

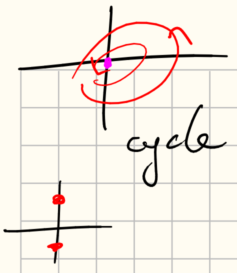
Principle: We can classify all possible qualitative behavior of linear systems; it depends only on the multiplicity of eigenvalues & location in $\mathbb{C}$

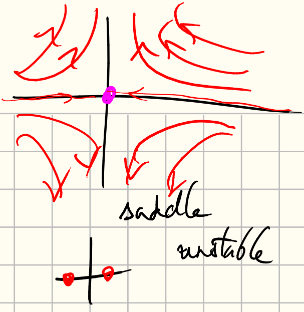
Eg: In 2D

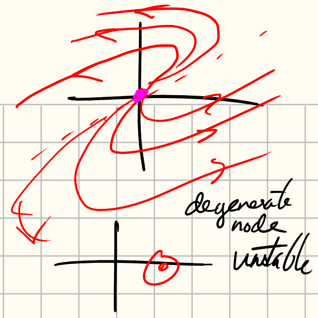


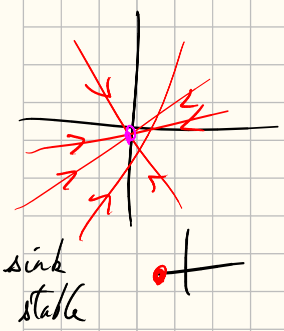
+ reflect across imaginary axis
same system, reversed time

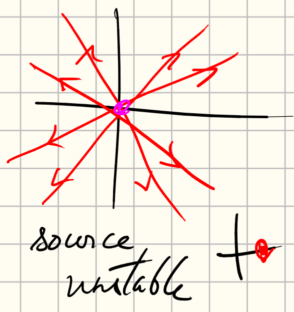










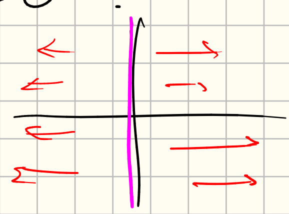


+ degenerate cases

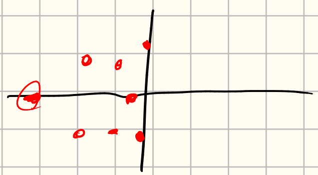
$\lambda = 0$

Eg:

linear



Conclude: A linear system is stable (solutions tend to the critical point or don't get further away)



if $Re(\lambda) \leq 0$ for all λ.

We can use this to study the qualitative behavior of non-linear systems. At critical points: (isolated & non-degenerate) the classification of qualitative behavior is the same.

Eg: Pendulum

$$\frac{d^2\theta}{dt^2} + \sin(\theta) = 0 \qquad \text{Non-linear}$$

First order system in θ $\dot{\theta}$

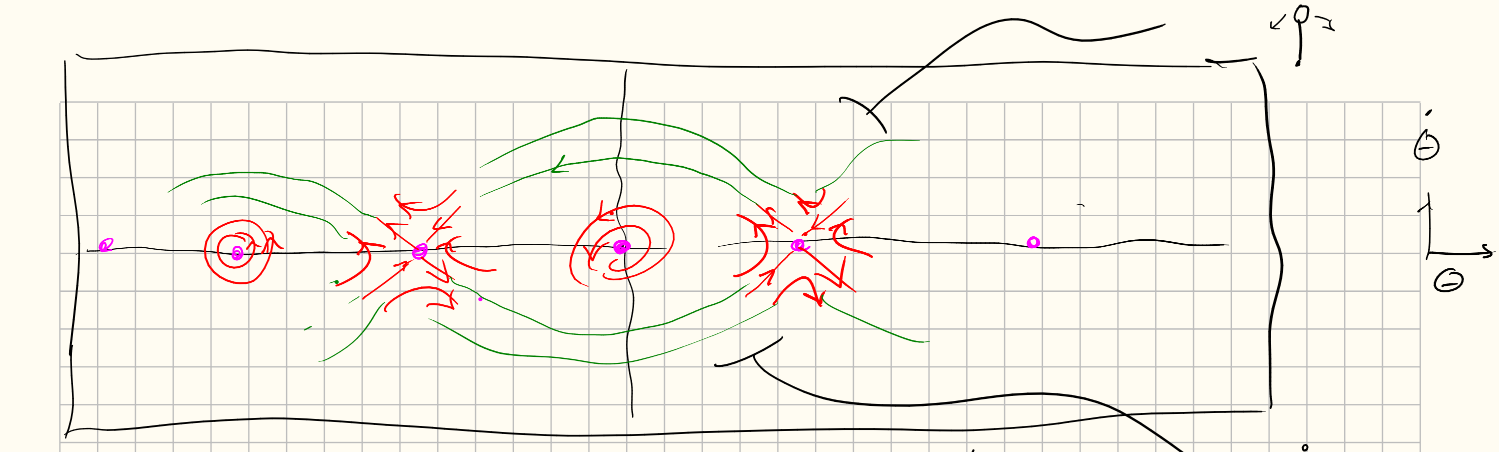
$$\begin{cases} \dfrac{d\theta}{dt} = \dot{\theta} \\ \dfrac{d\dot{\theta}}{dt} = -\sin(\theta) \end{cases}$$

– can't write as a matrix

Critical points: $\dfrac{d\vec{x}}{dt} = 0$ or $\dfrac{d\theta}{dt} = 0$, $\dfrac{d\dot{\theta}}{dt} = 0$

These occur at $\dot{\theta} = 0$ and $\sin(\theta) = 0$

$(\theta, \dot{\theta}) = (0,0), (\pi, 0), (2\pi, 0), \dots$ etc



Near each critical point the system is locally linear.

Near 0: $\sin(\theta) \sim \theta$.

$$\frac{d}{dt}\begin{pmatrix}\theta\\ \dot\theta\end{pmatrix} \sim \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\theta\\ \dot\theta\end{pmatrix} \quad \text{— cycle e.vals } \pm i$$

Near π: $\sin(\theta-\pi) = -(\theta-\pi)$

at $(\pi, 0)$
$$\frac{d}{dt}\begin{pmatrix}\theta-\pi\\ \dot\theta-0\end{pmatrix} \sim \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\theta\\ \dot\theta\end{pmatrix} \quad \text{— e.vals } +1, -1 \text{ - saddle}$$