ODE 5.20

Midterm review

Midterm due May 22 12:30 pm +24 hr grace period

§: First order ODEs

$$\frac{dy}{dt} = F(t,y)$$

Solution methods:

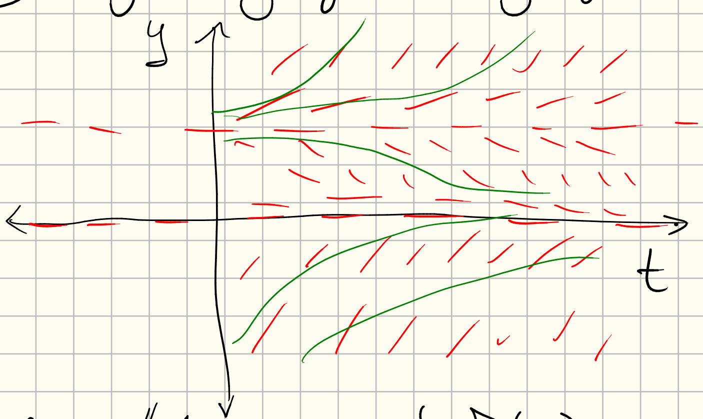
1. Graphical methods
   E.g. Slope field + connect dots to draw integral curves

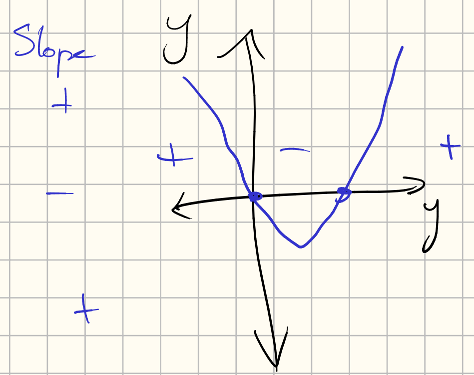
   Eg: $y' = y - x^2$ – study asymptotics using slope field

   Especially nice for autonomous systems

$$y' = F(y)$$

Eg: $y' = y^2 - y = y(y-1)$





Other methods solve $y' = F(t,y)$ in special cases

Eg: $y' = y^2 - y$ is separable i.e. $\frac{dy}{dt} = f(y)g(t)$

$$\int \frac{1}{f(y)}dy = \int g(t)dt$$

$$\frac{dy}{dt} = y^2 - y$$

$$\int \frac{1}{y^2-y}dy = \int dt$$

solve using partial fraction decomposition: subtle point about the integral

Eg: $\dfrac{dy}{dt} = y\sin(t) - 8\sin(t)$

separable $\dfrac{dy}{dt} = (y-8)\sin(t)$ $y \neq 8$

$$\int \frac{dy}{y-8} = \int \sin(t)\,dt$$

$$\log|y-8| = -\cos(t) + c$$

$$e^{\log|y-8|} = e^{-\cos(t)+c} = e^{c}e^{-\cos(t)} = Ae^{-\cos(t)}$$

$$|y-8| = Ae^{-\cos(t)}$$ – The RHS never changes sign

solution is

$y = -8 + Ae^{-\cos(t)}$ assuming $y-8>0$

otherwise

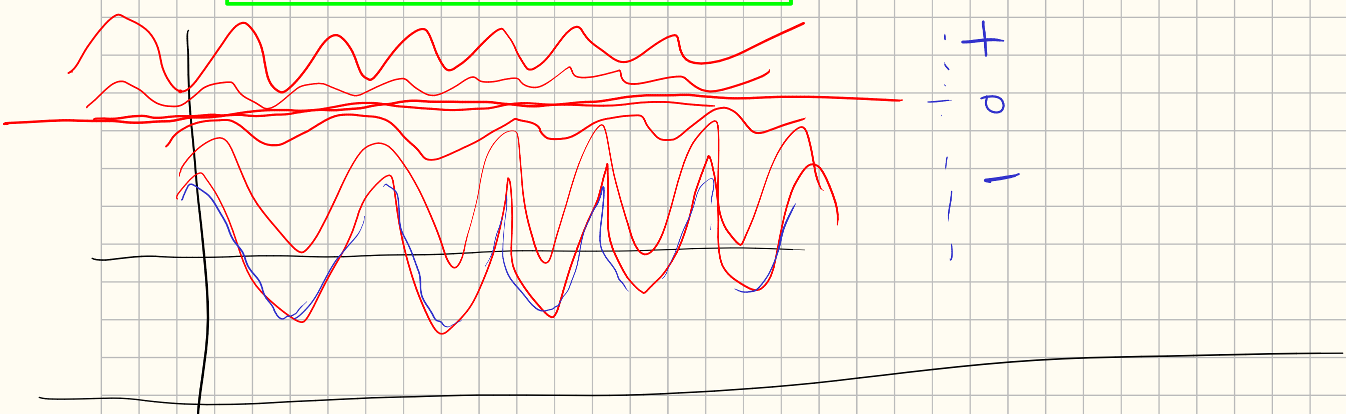
$-(y-8) = Ae^{-\cos(t)}$ let $B = -A$

$y = 8 + Be^{-\cos(t)}$

all possibilities are subsumed in the constant A.

$$\boxed{y = 8 + Ae^{-\cos(t)}}$$

$|y-8|$



First order linear equations

$y' = F(y,t)$ and $F(y,t) = -p(t)y + q(t)$

OR

$y' + p(t)y = q(t)$

this has a general solution using an integrating factor

- Multiply both sides by $\mu(t) = e^{\int p(t)dt}$

Eg: $ty' - y = t^2 e^{-t}$ $t > 0$

$y' - \frac{1}{t}y = te^{-t}$

IF: $e^{\int p(t)dt} = e^{-\int \frac{1}{t}dt}$

$= e^{-\log|t|}$

$t > 0$

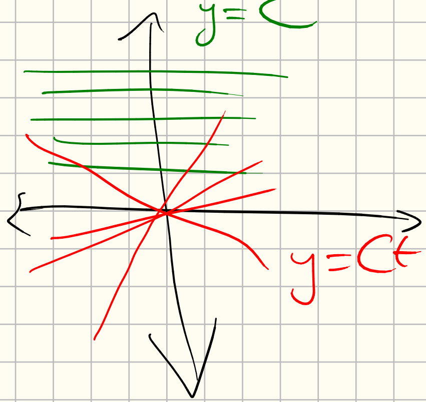
$e^{-\log(t)} = \frac{1}{t}$

$\frac{1}{t}y' - \frac{1}{t^2}y = e^{-t}$

$\frac{d}{dt}(\mu(t)y) = \frac{d}{dt}\left(\frac{1}{t}y\right) = e^{-t}$

$\frac{1}{t}y = \int e^{-t}dt = -e^{-t} + C$

$y = -te^{-t} + Ct$



2nd order equations

Focus on linear case

$$y'' + p(t)y' + q(t) = r(t)$$

Constant coefficient homogeneous case

$$y'' + ay' + b = 0$$

2 ways

gives e^{rt} works if $r^2 + ar + b = 0$

System of 1st order equations

$$\begin{pmatrix} y \\ \dot{y} \end{pmatrix}' = A \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

Eg: $y'' + 2y' + y = e^{-t}/1+t^2$

Inhomogeneous: find $\underbrace{y(t)}_{\text{complementary sol'n to } y''+2y'+y} + y_p(t)$ — any solution

$$\begin{cases} y' = \dot{y} & +0 \\ \dot{y}' = -2\dot{y} - y & + e^{-t}/1+t^2 \end{cases}$$

$\vec{y} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$ $\leadsto$ $\vec{y}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \vec{y} + \begin{pmatrix} 0 \\ e^{-t}/1+t^2 \end{pmatrix}$

Complementary solution:

$\vec{y}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \vec{y}$

$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{pmatrix} = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$

$\underline{\lambda = -1}$:

$A - \lambda I$ $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \vec{\xi} = 0$ $\vec{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Find generalized eigenvector

$\vec{\eta}$ such that

$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \vec{\eta} = \vec{\xi}$ $\vec{\eta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{y}_c(t) = c_1 e^{-t} \vec{\xi} + c_2 \left(t e^{-t} \vec{\xi} + e^{-t} \vec{\eta} \right)$ $\vec{\psi}_1 = e^{-t} \vec{\xi} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$

$\vec{\psi}_2 = \begin{pmatrix} te^{-t} \\ -te^{-t} \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}$

Package as

$$\Psi(t) = \left(\vec{\psi}_1(t) \,\middle|\, \vec{\psi}_2(t) \right) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{pmatrix} \vec{c}$$

$$y_p(t) = \Psi(t)\vec{u}(t)$$ – variation of parameters

$$\vec{u}(t) = \int_{t_0}^{t} \Psi^{-1}(s) g(s)\, ds \qquad g(s) = \begin{pmatrix} 0 \\ e^{-t}/_{1+t^2} \end{pmatrix}$$

$\Psi(s)$ is invertible : because we have a fundamental set of solutions

$$W[\vec{\psi}_1(t), \vec{\psi}_2(t)](t) = \det\begin{pmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t}+e^{-t} \end{pmatrix} = \det \Psi(t)$$

$$= -t\cancel{e^{-2t}} + e^{-2t} + (\cancel{t}e^{-2t}) = e^{-2t} \neq 0$$

$$\Psi^{-1}(s) = \frac{1}{W(s)}\begin{pmatrix} e^{-s}-se^{-s} & -se^{-s} \\ e^{-s} & e^{-s} \end{pmatrix} = \frac{1}{e^{-2s}}\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$$\frac{1}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\vec{u}(t) = \int_{t_0}^{t} e^{2s}\begin{pmatrix} e^{-s}-se^{-s} & -se^{-s} \\ e^{-s} & e^{-s} \end{pmatrix}\begin{pmatrix} 0 \\ e^{-s}/_{1+s^2} \end{pmatrix} ds$$

$$= \int_{t_0}^{t} \begin{pmatrix} 1-s & -s \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1/_{1+s^2} \end{pmatrix} ds$$

$$= \int_{t_0}^{t} \begin{pmatrix} -s/_{1+s^2} \\ 1/_{1+s^2} \end{pmatrix} ds = \begin{pmatrix} \int_{t_0}^{t} \frac{-s}{1+s^2} ds \\ \int_{t_0}^{t} \frac{1}{1+s^2} ds \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2}\log(s^2+1) \\ \tan^{-1}(t) \end{pmatrix}$$

<u>Solution</u>: $\vec{y} = \Psi(t)\vec{u}(t)$

$$\vec{y} = \Psi(t)\vec{c} + \begin{pmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t}+e^{-t} \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\log(t^2+1) \\ \tan^{-1}(t) \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

$$\boxed{y(t) = c_1 e^{-t} + c_2 te^{-t} + e^{-t}\left(-\frac{1}{2}\log(t^2+1)\right) + \tan^{-1}(t)\cdot te^{-t}}$$

Rewriting nth order as 1st order systems:

Eg: $y''' + ay'' + by' + cy = g(t)$

$$\vec{y} = \begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \end{pmatrix} \qquad \begin{cases} y' = \dot{y} \\ \dot{y}' = \ddot{y} \\ \ddot{y}' = -a\ddot{y} - b\dot{y} - cy + g(t) \end{cases}$$

$$\begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \end{pmatrix}' = \begin{array}{c|ccc} & y & \dot{y} & \ddot{y} \\ \hline y' & 0 & 1 & 0 \\ \dot{y}' & 0 & 0 & 1 \\ \ddot{y}' & -c & -b & -a \end{array} \begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ g(t) \end{pmatrix}$$

Analyze the asymptotic behavior of the 2nd order ODE

$$y'' - 4y' + 4y = 0 \qquad \text{(HW\#3 Problem 4)}$$

For which initial conditions y_0, y_0' does $y(t) \to +\infty$, $\to -\infty$

$$\vec{y} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

$$\begin{pmatrix} y \\ \dot{y} \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -4 & +4 \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

$\underline{\lambda = 2}$: $A - 2I = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}$ $\quad \vec{\xi} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a 2-eigenvector

$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \vec{\eta} = \vec{\xi}$ has solution $\vec{\eta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

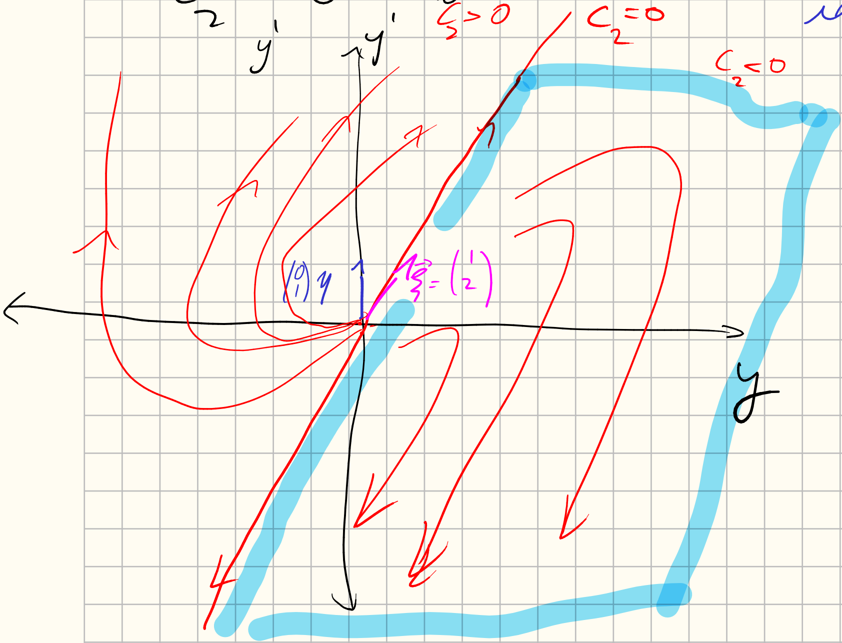
$$\begin{pmatrix} y \\ \dot{y} \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left(t e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$y = c_1 e^{2t} + c_2 t e^{2t}$$

- One solution: solve c_1 & c_2 in terms of y_0, y_0'

$C_2 > 0$ solution $\to +\infty$

$C_2 < 0$ solution $\to -\infty$



in blue region $y \to -\infty$

If $\begin{bmatrix} y \\ y' \end{bmatrix}$ below $y_0' = 2y_0$

$y_0' < 2y_0$ then

$y(t) \to -\infty$ at $t \to \infty$

$y_0' > 2y_0$

$y(t) \to \infty$ at $t \to \infty$