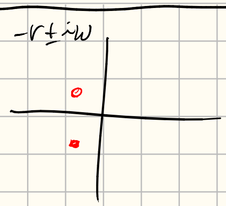
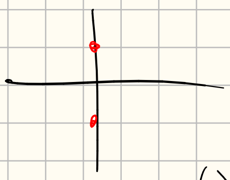
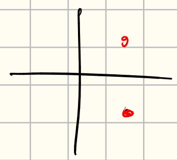
ODEs 5-24

$-r \pm i\omega$

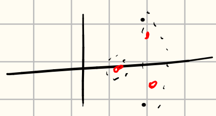
$e^{-rt}(a\cos(\omega t)+b\sin(\omega t)) \to 0$ $a\cos(\omega t)+b\sin(\omega t)$ $e^{rt}(a\cos(\omega t)+b\sin(\omega t))$

$r>0$ $\parallel$

$A\cos(\omega t+s)$

Hint for P3 on HW#5: Make matrices block matrices

Eg:



$$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & 1 & 0 \\ \hline 0 & 0 & 1/2 \end{array}\right)$$

We've covered constant coefficient homogeneous linear systems.

From here:
- Non-constant coefficients
- inhomogeneous term gets badly behaved
- Non-linear equations

Local qualitative behavior of autonomous systems

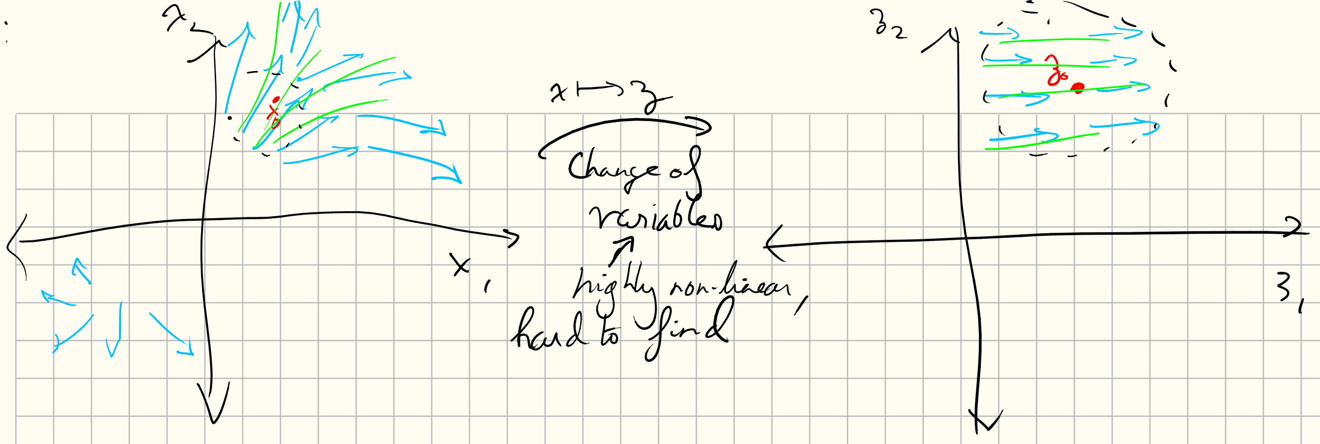
Eg: $\begin{cases} \dfrac{dy}{dt} = F_y(x,y) \\ \dfrac{dx}{dt} = F_x(x,y) \end{cases}$ $\longrightarrow$ how do we study this?

Theorem (Behavior near non-critical points)

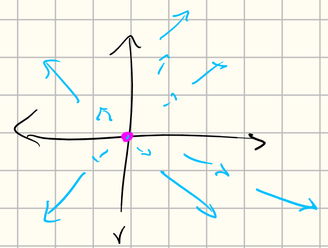
Given some autonomous system $x(t)' = F(x(t))$ $F\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} F_1(\vec{x}) \\ \vdots \\ F_n(\vec{x}) \end{pmatrix}$

if $F(x)$ is continuous and $F(x_0) \neq \vec{0}$ then around x_0 nearby trajectories travel roughly parallel to each other.

Rmk:



What about when $F(x_0) = \vec{0}$?



There are <u>critical points</u> of the system

$$\vec{x}' = F(\vec{x}).$$

Eg: $\vec{x}' = A\vec{x}$

$F(\vec{x}) = A\vec{x}$

$F(\vec{0}) = \vec{0}$

Thm (Qualitative behavior near critical points), learn soon

Assume x_{crit} is a <u>isolated</u> & <u>non-degenerate</u> critical point of $F(\vec{x})$ and $F(\vec{x})$ is locally linear $\left(F(\vec{x}-\vec{x}_0) = J(\vec{x}-\vec{x}_0) + O(|x-x_0|) \right)$

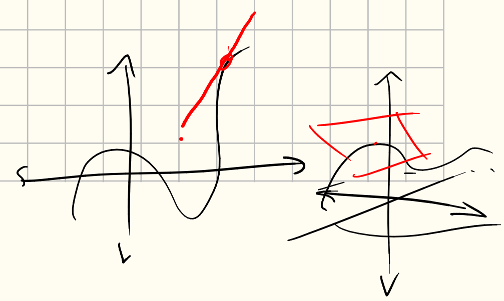
Then the qualitative behavior near x_{crit} is equivalent to the behavior of a linear system near x_{crit}.

$$(\vec{X}-\vec{X}_{crit})' = J_F(\vec{x}-\vec{x}_{crit}) \quad \text{where } J_F \text{ is the Jacobian of } F$$

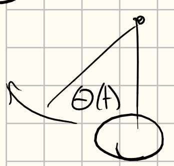
Def: The Jacobian of $F = \begin{pmatrix} F_1(\vec{x}) \\ \vdots \\ F_n(\vec{x}) \end{pmatrix}$ is the matrix

$$J_F(\vec{x}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \ddots & & \\ \vdots & & \ddots & \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & & \frac{\partial F_n}{\partial x_n} \end{pmatrix}$$

Rmk: $J_F(\vec{x})$ is the derivative of $\vec{F}$ at $\vec{x}$.



Eg: Damped pendulum



damping parameter $c \geq 0$

$$\theta'' + c\theta' + \sin(\theta) = 0$$

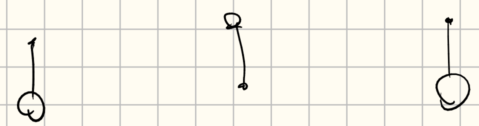
Step 1: Find critical points

$$\theta' = \dot{\theta}$$
$$\dot{\theta}' = -c\dot{\theta} - \sin(\theta)$$

$$\vec{\Theta}' = F(\vec{\Theta}) \qquad F = \begin{pmatrix} \dot{\theta} \\ -c\dot{\theta} - \sin(\theta) \end{pmatrix} \quad \begin{matrix} F_1(\theta,\dot{\theta}) \\ F_2 = F_2(\theta,\dot{\theta}) \end{matrix}$$

$$F(\vec{\Theta}) = 0 \qquad \begin{pmatrix} \dot{\theta} \\ -c\dot{\theta} - \sin(\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} \dot{\theta} &= 0 \\ 0 - \sin(\theta) &= 0 \\ \sin(\theta) &= 0 \end{aligned}$$

Critical points: $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ \pi \end{pmatrix} \quad \begin{pmatrix} 0 \\ 2\pi \end{pmatrix} \quad \dots \quad \begin{pmatrix} 0 \\ k\pi \end{pmatrix} \quad k \in \mathbb{Z}$



Step 2: Find $J_F(\vec{x}_{crit})$ for each $\vec{x}_{crit}$

$$\vec{\Theta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \left(\text{or } \begin{pmatrix} 0 \\ 2k\pi \end{pmatrix}\right) \qquad F = \begin{pmatrix} \dot{\theta} \\ -c\dot{\theta} - \sin(\theta) \end{pmatrix} = \begin{pmatrix} F_1(\theta,\dot{\theta}) \\ F_2(\theta,\dot{\theta}) \end{pmatrix}$$

$$J_F(\theta) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos\theta & -c \end{pmatrix}$$

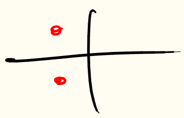
$x_1 = \theta, \ x_2 = \dot{\theta}$

at $\theta = \begin{pmatrix} 0 \\ 2k\pi \end{pmatrix}$ | at $\theta = \begin{pmatrix} 0 \\ (2k+1)\pi \end{pmatrix}$

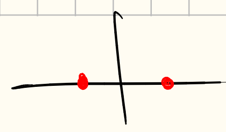
$$J_F(\vec{\theta}_{crit}) = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix} \qquad J_F(\vec{\theta}_{crit}) = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$$

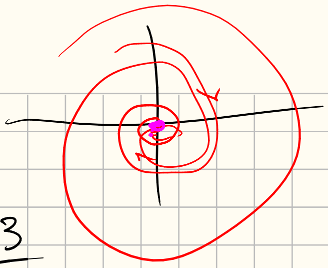
$$|J_F - \lambda I| = (-\lambda)(-c-\lambda) + 1 = \lambda^2 + \lambda c + 1$$

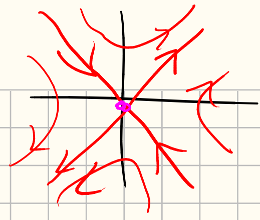
$$\lambda = \frac{-c \pm \sqrt{c^2-4}}{2}$$



$$\det|J_F - \lambda I| = (-\lambda)(-\lambda - c) - 1 = \lambda^2 + c\lambda - 1 = \frac{-c}{2} \pm \frac{1}{2}\sqrt{c^2+4}$$

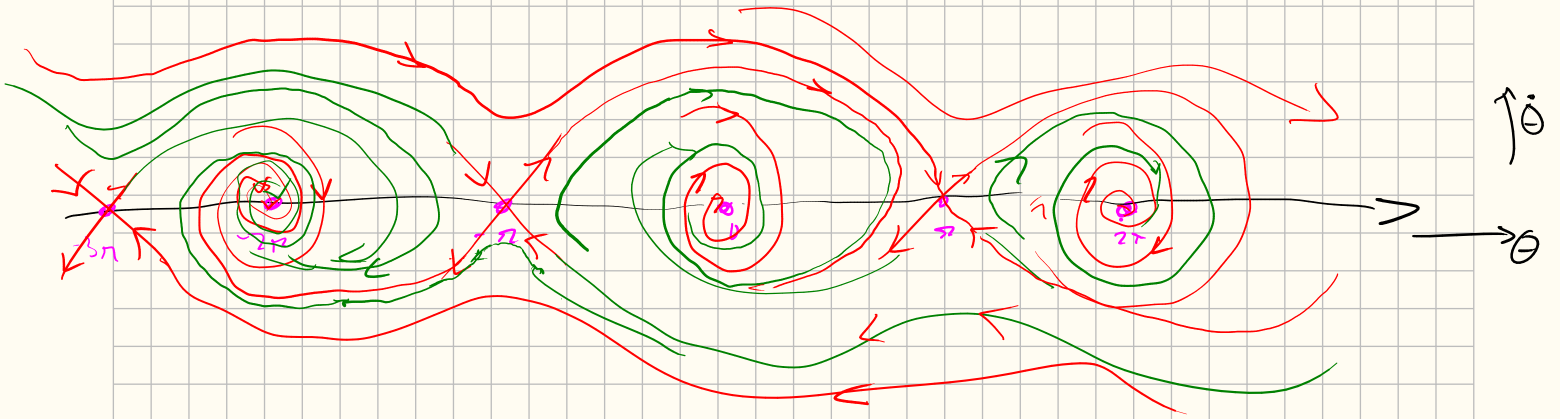






Step 3

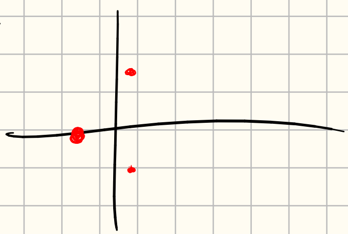
Phase diagram by connecting the dots



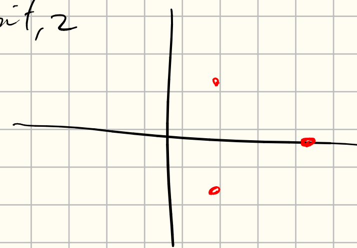
Global behavior: Can be very difficult / impossible to deduce from local behavior.

Eg: Rössler system   non-linear chaotic system w/ 2 critical points whose global behavior is hard to deduce from local behavior: see demo

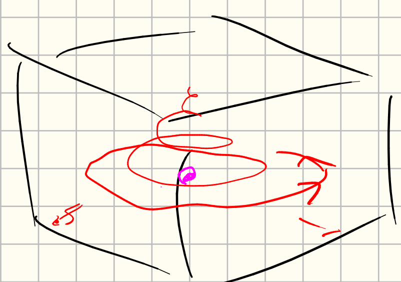
$x_{crit,1}$

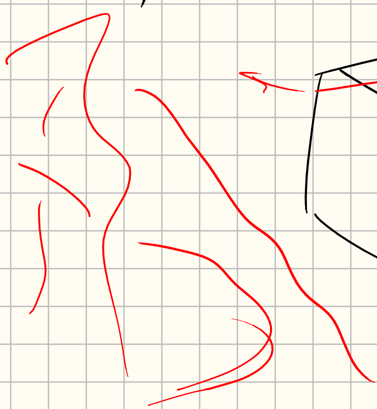


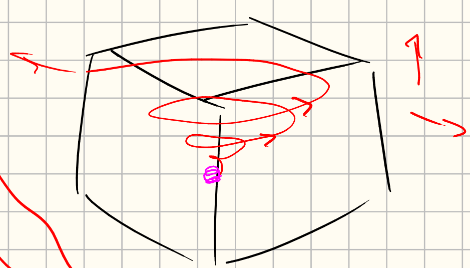
$x_{crit,2}$



Chaos







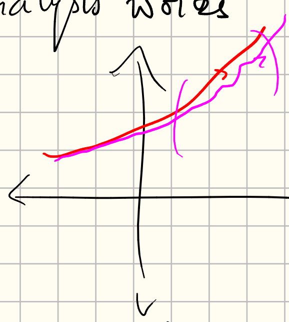
Def: A critical point $\vec{x}_{crit}$ is non-degenerate if all eigenvalues of $J_F(\vec{x}_{crit})$ are non-zero

(Or $\det |J_f(\vec{x}_{crit})| \neq 0$).

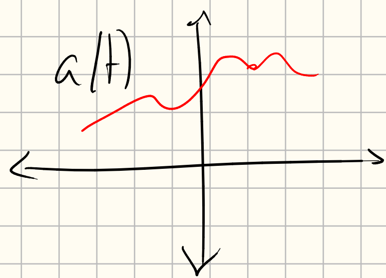
What about non-autonomous systems?

$$\vec{x}(t)' = F(\vec{x}, t)$$

If $F(\vec{x}, t)$ is slowly varying with $t \Rightarrow$ basically the same analysis works



$$y' = a(t)y$$



What if $a(t)$ → changes from + to − ?
→ $\to \infty$ at $t \neq \infty$

Try to understand this for 2nd order equations (1d case, we have a solution)

Eg: $y'' + \frac{\alpha}{t} y' + \frac{\beta}{t^2} y = 0$ has a singularity at $t = 0$

Method: Take Taylor series expansions of the coefficients $a(t)$ & solutions $y(t)$

Power series review

Def: A power series is an expression

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

We add/multiply/take derivatives or integrals just like polynomials

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n + \sum_{m=0}^{\infty} b_m (x - x_0)^m = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n$$

$$\frac{d}{dx} \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n.$$

Def:
- $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges at x if $\lim_{N\to\infty} \sum_{n=0}^{N} a_n (x-x_0)^n$ converges.
- $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely at x if $\lim_{N\to\infty} \sum_{n=0}^{N} |a_n (x-x_0)^n|$ converges

Proposition: (Ratio test) Suppose
$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = |x-x_0| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-x_0| L$$

If
- $|x-x_0| L < 1$ – $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely
- $|x-x_0| L > 1$ – $\sum a_n (x-x_0)^n$ diverges

Def: The radius of convergence of $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is largest value R s.t. if $|x-x_0| < R$ then the series converges at x.

Eg: Taylor series let $f(x)$ be a function w/ all derivatives (ie. smooth)

Then $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$ is the Taylor series for $f(x)$ at x_0.

Eg: $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$. Radius of convergence is ∞

(b/c $\lim_{n\to\infty} \left| \frac{\frac{1}{n!}}{\frac{1}{(n-1)!}} \right| = 0$

Eg: $\log(x)$ at $x=1$ $\quad f(1)=0 \quad f'(1) = \frac{1}{1} = 1 \quad f''(1) = \frac{-1}{1^2} = -1$

$$f^{(n)}(1) = (-1)^{n-1} \cdot (n-1)!$$

$$\log(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} (x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n$$

Radius of convergence 1.