ODEs 5-25

Goal: Using power series to solve ODEs.

Eg: Imagine we knew nothing about e^x but wanted to study natural growth

$$\frac{dy}{dt} = ky \qquad y(0) = A$$

We can use the equation to find $y'(0)$:

$y'(0) = kA$. We can get more

$$y'' = \frac{d}{dt}(y') \underset{\text{use eq.}}{=} \frac{d}{dt}(ky) = ky'$$

$$y''(0) = ky'(0) = k(kA) = k^2 A$$

$$y^{(n)}(0) = ky^{(n-1)}(0) = k^n A.$$

We can recover $y(t)$ from $y^{(n)}(t_0)$ for all n and a single t_0 for a nice function $y(t)$.

$$y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} t^n = \sum_{n=0}^{\infty} \frac{k^n A}{n!} t^n$$

$$= A \sum_{n=0}^{\infty} \frac{1}{n!}(kt)^n =: Ae^{kt}$$

Eg: $y'' + y = 0$

Method: Assume y has a power series representation (i.e. y is analytic (= equal to its Taylor series))

Ansatz: $y(t) = \sum_{n=0}^{\infty} a_n t^n$

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots + a_n t^n + \dots$$

$$y' = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \dots + (n+1)a_{n+1} t^n + \dots$$

$$+\ y'' = 2a_2 + 3\cdot 2 a_3 t + 4\cdot 3 a_4 t^2 + \dots + (n+2)(n+1)a_{n+2} t^n + \dots$$

$$y'' + y = a_0 + 2a_2 + (a_1 + 3\cdot 2 a_3)t + (a_2 + 4\cdot 3 a_4)t^2 + \dots + (a_n + (n+2)(n+1)a_{n+2})t^n$$

$$= 0 + 0t + 0t^3 + \dots + 0t^n + \dots$$

We want to find ∞ number of a_n's

We have ∞ number of equations

Our equations are of the form

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$ Once we know a_0 and a_1

we can solve $a_2, a_3, a_4, a_5, \ldots\ldots$

$$a_{2k} = \frac{-a_{2k-2}}{(2k)(2k-1)} = \frac{(-1)^2 a_{2k-4}}{(2k)(2k-1)(2k-2)(2k-3)} = \cdots$$

$$= \frac{(-1)^k a_0}{(2k)!}.$$

$$a_{2k+1} = \frac{-a_{2k-1}}{(2k+1)(2k)} = \frac{(-1)^2 a_{2k-3}}{(2k+1)(2k)(2k-1)(2k-2)}$$

$$= \frac{(-1)^k a_1}{(2k+1)!}$$

General solution:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n = \sum_{k=0}^{\infty} a_{2k} t^{2k} + \sum_{k=0}^{\infty} a_{2k+1} t^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k a_0}{(2k)!} t^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k a_1}{(2k+1)!} t^{2k+1}$$

$$= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}$$

$$= a_0 \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= a_0 \cos(t) + a_1 \sin(t)$$

Def: A recurrence relation is a system of equations of the form

$$a_n = F(a_{n-1}, \ldots, a_{n-k}) \qquad n = 0, 1, \ldots$$

The order is k

Fact: There is a unique solution to a kth order recurrence relation with initial data $a_0, \ldots, a_{k-1}$

Eg: Fibonacci sequence

$$F_n = F_{n-1} + F_{n-2} \qquad F_0 = 1, \; F_1 = 1$$

is a 2nd order recurrence relation

$$F_0 = 1, \; F_1 = 1, \; F_2 = 1+1 = 2, \; F_3 = 2+1 = 3, \ldots$$

Eg: Solve the IVP

$$y'' = y' + y \qquad y(0) = 1 \quad y'(0) = 1$$

and calculate $y^{(n)}(0)$ (We'll use summation notation, but we could have kept the same notation as before)

$$y(t) = \sum_{n=0}^{\infty} a_n t^n \qquad y'(t) = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n \qquad y''(t) = \sum (n+2)(n+1) a_{n+2} t^n$$

$-y - y' + y'' = 0$ becomes

$$-\sum_{n=0}^{\infty} a_n t^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n = 0$$

$$-a_n - (n+1) a_{n+1} + (n+2)(n+1) a_{n+2} = 0$$

$$a_{n+2} = \frac{a_n + (n+1) a_{n+1}}{(n+2)(n+1)} = \frac{a_n}{(n+2)(n+1)} + \frac{a_{n+1}}{n+2}$$

Set $\tilde{a}_n = n! \, a_n$

$$\frac{\tilde{a}_{n+2}}{(n+2)!} = \frac{\tilde{a}_n}{n!(n+1)(n+2)} + \frac{\tilde{a}_{n+1}}{(n+1)!(n+2)} \qquad \frac{\tilde{a}_{n+2}}{(n+2)!} = \frac{\tilde{a}_n}{(n+2)!} + \frac{\tilde{a}_{n+1}}{(n+2)!}$$

$$\tilde{a}_{n+2} = \tilde{a}_n + \tilde{a}_{n+1}$$

$$y(t) = \tilde{a}_0 + \tilde{a}_1 t + O(t^2) \Rightarrow \tilde{a}_0 = y(0) \quad \tilde{a}_1 = y'(0)$$

Our IVP says $y(0) = 1$ $y'(0) = 1$

Our recurrence relation is

$$\tilde{a}_{n+2} = \tilde{a}_n + \tilde{a}_{n+1}, \; \tilde{a}_0 = 1, \; \tilde{a}_1 = 1 \Rightarrow \tilde{a}_n = F_n \quad \text{- Fibonacci number}$$

Solution is $y(t) = \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} \frac{F_n}{n!} t^n$

where F_n is the n-th Fibonacci number.

$$y^{(n)}(0) = F_n$$

We could have solved our equation using other methods:

$$y'' - y' - y = 0$$

Char. poly. $r^2 - r - 1 = 0$ with roots $\underbrace{\frac{1+\sqrt{5}}{2}}_{=\varphi}$, $\underbrace{\frac{1-\sqrt{5}}{2}}_{1-\varphi}$

$$y(t) = c_1 e^{\varphi t} + c_2 e^{(1-\varphi)t}.$$

$y(0) = 1$ $\qquad c_1 + c_2 = 1$

$y'(0) = 1$ $\qquad \varphi c_1 + (1-\varphi) c_2 = 1$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ \varphi & 1-\varphi & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1-2\varphi & 1-\varphi \end{array}\right)$$

$$c_2 = \frac{1-\varphi}{1-2\varphi} \qquad c_1 = 1 - \frac{1-\varphi}{1-2\varphi}$$

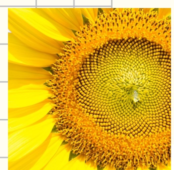
$$c_1 \,\&\, c_2 \neq 0$$

Therefore $y(t) \sim c_1 e^{\varphi t}$ because $\varphi > 1-\varphi$.

Exercise

It follows that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{\left.\frac{d^{n+1}}{dt^{n+1}}\right|_{t=0} e^{\varphi t}}{\left.\frac{d^n}{dt^n}\right|_{t=0} e^{\varphi t}} = \varphi \text{ - Golden ratio}$$




Now: Non-constant coefficients

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0$$

Frequently: 1) these equations arise as Newtonian mechanical systems

2) $P(t) = 0$ at some time t_0 $\Rightarrow$ we can't solve for $\frac{d^2y}{dt^2}$ without our equation blowing up at t_0.

Eg: $y'' + \frac{Q(t)}{P(t)}y' + \frac{R(t)}{P(t)}y = 0 \rightsquigarrow y''(t_0) + \infty y'(t_0) + \infty y(t_0) = 0$

Today: We'll focus on the case where this doesn't happen

Def: t_0 is an <u>ordinary point</u> of the ODE

$$P(t)\frac{d^2y}{dt^2}+Q(t)\frac{dy}{dt}+R(t)y=0$$

if $P(t_0)\neq 0$

If t_0 is an ordinary point then near t_0 our equation becomes

$$\frac{d^2y}{dt^2}+\frac{Q(t)}{P(t)}\frac{dy}{dt}+\frac{R(t)}{P(t)}y=0$$

We can imagine that given $y(t_0)$ & $y'(t_0)$ we could use the equation to solve for $y''(t_0)$, $y^{(3)}(t_0)$, $y^{(4)}(t_0)$, ... and get a solution in terms of Taylor series $y(t)=\sum_{n=0}^{\infty}\frac{y^{(n)}}{n!}(t-t_0)^n$.

Rmk: This doesn't work if $P(t_0)=0$. We can't even find $y''(t_0)$:

because $P(t_0)y''(t_0)+Q(t_0)y'(t_0)+R(t_0)y(t_0)=0$

$$0+Q(t_0)y'+R(t_0)y=0$$

Thm: If • t_0 is an ordinary point of

$$P(t)y''+Q(t)y'+R(t)y=0$$

• $\frac{Q(t)}{P(t)}$ and $\frac{R(t)}{P(t)}$ are <u>analytic</u> around t_0 with radius of convergence $\geq R$

equal to Taylor series

Then $y(t)$ has solution $\sum_{n=0}^{\infty}a_n(t-t_0)^n$ with radius of convergence $\geq R$.

Eg: Solve $(1-x^2)y''-xy'+\alpha^2y=0$ | Where can $\frac{x}{1-x^2}$ & $\frac{\alpha^2}{1-x^2}$ not have convergent power series representations?

near $x=0$. We only expect a solution for $x\in(-1,1)$.

This is the Chebyshev equation

Ansatz $y(x)=\sum_{n=0}^{\infty}a_nx^n$ $\quad y'(x)=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$

$$y''(x)=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(1-x^2)\left(\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n\right)-x\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n\right)+\alpha^2\left(\sum_{n=0}^{\infty}a_nx^n\right)=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\alpha^2\sum_{n=0}^{\infty}a_nx^n=0$$

$$n'=n+2 \qquad n'=n+1$$

$$\sum_{n}(n+2)(n+1)a_{n+2}x^n-\sum_{n'=2}^{\infty}(n')(n'-1)a_{n'}x^{n'}-\sum_{n'=1}^{\infty}n'a_{n'}x^{n'}+\alpha^2\sum_{n=0}^{\infty}a_nx^n=0$$

$$(n+2)(n+1)a_{n+2}-n(n-1)a_n-na_n+\alpha^2a_n=0$$

$$a_{n+2}=\frac{(n(n-1)+n-\alpha^2)a_n}{(n+2)(n+1)}$$

$$a_{n+2}=\frac{(n-\alpha)(n+\alpha)}{(n+2)(n+1)}a_n$$

$$a_2=\frac{-\alpha^2}{2}a_0 \qquad a_3=\frac{(1-\alpha)(1+\alpha)}{3\cdot 2}a_1$$

$$a_{2\ell+2}=\prod_{k=1}^{\ell}\frac{(2k-\alpha)(2k+\alpha)}{(2k)(2k-1)}a_0=a_0\left[\frac{(2\ell-\alpha)(2\ell+\alpha)(2\ell-2-\alpha)(2\ell-2+\alpha)\dots(2-\alpha)(2+\alpha)}{(2\ell+2)!}\right]$$

$a_{2\ell+1}$ has a similar formula $=a_1\left[\dfrac{\prod_{k=1}^{\ell-1}(2k+1-\alpha)(2k+1-\alpha)}{(2\ell+1)!}\right]$

Solutions are

$$y(x)=a_0\sum_{n=0}^{\infty}\left[\quad\right]x^n+a_1\sum_{n=0}^{\infty}\left[\quad\right]x^n$$

$$\boxed{y(x)=a_0\sum_{\ell=0}^{\infty}\frac{\prod_{k=1}^{\ell-1}(2k-\alpha)(2k+\alpha)}{(2\ell)!}x^{2\ell}+a_1\sum_{\ell=0}^{\infty}\frac{\prod_{k=1}^{\ell-1}(2k+1+\alpha)(2k+1-\alpha)}{(2\ell+1)!}x^{2\ell+1}}$$