ODEs 5-26

Recall: Power series solution method:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + O(x^4)$$

works for small x

Works for nth order equations / systems of equations

Eg: $\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} a_n x^n \\ \sum_{n=0}^{\infty} b_n x^n \end{pmatrix}$

Eg. Chebyshev equation

$$(1-x^2)y'' + xy' + \alpha^2 y = 0$$

Solution:
- $y = \sum_{n=0}^{\infty} a_n x^n$ $\quad$ a_0 = arbitrary, a_1 = arbitrary

  $-1 < x < 1$

- $$a_{n+2} = \frac{(n-\alpha)(n+\alpha)}{(n+2)(n+1)} a_n$$

- $$y(x) = a_0 \sum_{\ell=0}^{\infty} \frac{\prod_{k=0}^{\ell-1} (2k-\alpha)(2k+\alpha)}{2\ell!} x^{2\ell} + a_1 \sum_{\ell=0}^{\infty} \frac{\prod_{k=0}^{\ell-1} (2k+1-\alpha)(2k+1+\alpha)}{(2\ell+1)!} x^{2\ell+1}$$

$n = \alpha \Rightarrow a_{n+2} = \underset{\sim}{0} \cdot a_n = 0 \Rightarrow a_{n+4} = a_{n+6} = a_{n+8} = 0$

$\Rightarrow$ We get polynomial solutions

$\alpha = 1$: $\quad a_1 = 1 \quad a_0 = 0 \quad . \quad a_3 = \frac{(1-\alpha)(1+\alpha)}{(1+2)(1+1)} = 0 \quad a_n = 0$ if $n \neq 1$

$\boxed{y(x) = 1}$ is a solution.

$\alpha = 2$: $\quad a_1 = 0, \quad a_0 = 1 \quad a_{2k-1} = 0$

$$a_2 = \frac{(n-\alpha)(n+\alpha)}{(n+2)(n+1)} a_0 = \frac{(0-2)(0+2)}{(0+2)(0+1)} a_0 = -2$$

$$a_4 = \frac{\underset{\sim}{(2-\alpha)}}{\underset{\sim\sim}{\quad}} a_2 = \underset{\sim}{0} \cdot a_2 = 0 \Rightarrow a_{4+2k} = 0$$

$\boxed{y(x) = -2x^2 + 1}$ is also a solution.

Rmk: Let $T_n(x)$ be the polynomial solution for

$$\begin{cases} a_0 = (-1)^{n/2} \quad a_1 = 0 & n \text{ even} \\ a_1 = (-1)^{n-1/2} n \quad a_0 = 0 & n \text{ odd} \end{cases}$$

then $T_n(x)$ is the Chebyshev polynomial

these have some remarkable properties

Eg: $T_n(\cos\theta) = \cos(n\theta)$ $\quad n = 2$

$$T_2(x) = 2x^2 - 1 \Rightarrow \cos(2\theta) = 2\cos^2(\theta) - 1$$

Thm (This ansatz works for ordinary points)

Let $P(x) = 1 + \sum_{n=1}^{\infty} p_n (x - x_0)^n$

$$Q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n$$

$$R(x) = \sum_{n=0}^{\infty} r_n (x - x_0)^n$$

Then this method produces the general solution to

$$P(x) y'' + Q(x) y' + R(x) y = 0$$

of the form $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$.

Rmk: For $(x - x_0)$ small we can approximate by taking

$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots \; O(x^n)$

$P(x) = 1 + p_1 x + \cdots + O(x^n)$ $\quad$ etc.

What if $P(x_0) = 0$? We can't solve for y'' in terms of y, y'

$y^{(3)}$ ——— $, y''$

In general: We can't do anything

But for some cases, we have a method

↑ regular singular points

Def: ∘ A *singular point* of the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$ is

a point x_0 such that $P(x_0) = 0$.

∘ A *regular singular point* is a singular point where

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} \text{ exists} \qquad \text{AND}$$

$$\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} \text{ exists}$$

Eg: (Prototype for regular singular points) The *Euler equation*

$$x^2 y'' + \alpha x y' + \beta y = 0$$ has a regular singular point at $x = 0$. α, β are constants

Check: $Q(x) = \alpha x$, $R(x) = \beta$, $P(x) = x^2$

$$\lim_{x \to 0} x \frac{Q(x)}{P(x)} = \lim_{x \to 0} x \frac{\alpha x}{x^2} = \alpha \quad \checkmark$$

$$\lim_{x \to 0} x^2 \frac{R(x)}{P(x)} = \lim_{x \to 0} x^2 \frac{\beta}{x^2} = \beta \quad \checkmark$$

Fact: If we let $\alpha = \alpha(x)$ and $\beta = \beta(x)$ be power series which are well defined at $x = x_0$ then this is "the only example"

$$\frac{Q(x)}{P(x)} = p(x) \qquad \frac{R(x)}{P(x)} = q(x)$$

So our equation is

$$y'' + p(x)y' + q(x)y = 0$$ our condition for a regular singular point (@0) becomes $\lim_{x \to 0} x p(x)$ and $\lim_{x \to 0} x^2 q(x)$ exist

Claim All examples are of the form

$$x^2 y'' + x\underbrace{(p_0 + p_1 x + p_2 x^2 + \cdots)}_{\text{"}\alpha\text{"}} y' + \underbrace{(q_0 + q_1 x + q_2 x^2 + \cdots)}_{\text{"}\beta\text{"}} y = 0$$

$$p(x) = \frac{x \sum_{n=0}^{\infty} p_n x^n}{x^2} \qquad q(x) = \frac{1}{x^2}\left(\sum_{n=0}^{\infty} q_n x^n\right)$$

$$\lim_{x \to 0} x p(x) = \frac{x^2}{x^2}\sum_{n=0}^{\infty} p_n x^n = p_0 \quad \checkmark \quad \lim_{x \to 0} x^2 q(x) = \frac{x^2}{x^2}\sum q_n x^n = q_0 \quad \checkmark$$

Recall: Euler equation has solution $c_1 e^{\log(x) r_1} + c_2 e^{\log(x) r_2}$ where r_1 & r_2 are roots of of a characteristic polynomial of a equivalent linear equation (HW#3)

OR $c_1 x^{r_1} + c_2 x^{r_2}$ r_1 & r_2 are not necessarily $\in \mathbb{Z}$.

Solution to $x^2 y'' + x p(x) y' + q(x) y = 0$ — general equation w/ regular singular point @ 0

Provided by the Frobenius ansatz

If r_1 & r_2 are roots of the indicial equation

$$r(r-1) + p(0) r + q(0) \qquad r_1 - r_2 \notin \mathbb{Z}$$

then $y(x) =$

$$x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

provides a solution to the equation $x^2 y'' + x p(x) y' + q(x) y = 0$

Why does this work? Answer: guess $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ lowest order terms of the ODE $\Rightarrow$ indicial equation

$$x^2 y'' + x p(x) y' + q(x) y = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

LHS: $x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) + x p(x) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right)$

$$+ q(x) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + p(x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + q(x) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$\downarrow x=0$

$$r(r-1) a_0 + p(0) r a_0 + q(0) a_0 = 0$$

Indicial equation: $r(r-1) + p(0) r + q(0) = 0$ must be true.

Remarkably: This is all we need: all of the a_n can be determined from the first 2 if $r_1 - r_2$ (roots of indicial equation) is not an integer.

Eg: Euler equation

$$x^2 y'' + x\alpha y' + \beta y = 0$$

$$p(x) = \alpha \qquad (= p(0))$$
$$q(x) = \beta \qquad (= q(0))$$

Indicial equation
$$r(r-1) + p(0) r + q(0) = 0$$
$$r^2 - r + \alpha r + \beta = 0 \qquad \text{, with roots } r_1 \,\&\, r_2$$

Solutions are
$$y(x) \; c_1 x^{r_1} + c_2 x^{r_2} \qquad c_1 \,\&\, c_2 \in \mathbb{C}$$

Eg: $y(x) = \sin(\log(x))$ if $r_1 = i$, $r_2 = -i$.

Hypergeometric equation

A huge variety of 2nd order ODEs with regular singular points can be written as a hypergeometric equation after a simple change of variables.

The hypergeometric equation is the 2nd order linear ODE

$$x(1-x) y'' + (\gamma - (1+\alpha+\beta)x) y' - \alpha\beta y = 0$$

It has regular singular points @ 0, 1, and ∞

$$P(x) = x(1-x)$$
$$Q(x) = \gamma - (1+\alpha+\beta)x$$
$$R(x) = -\alpha\beta$$

Today: Solutions @ $x=0$

Check regular singular point

$$\lim_{x\to 0} x \frac{Q(x)}{P(x)} = \lim_{x\to 0} x \frac{(\gamma - (1+\alpha+\beta)x)}{x(1-x)} = \frac{x\gamma - (1+\alpha+\beta)x^2}{x - x^2} = \gamma = p(0) \checkmark$$

$$\lim_{x\to 0} x^2 \frac{R(x)}{P(x)} = \lim_{x\to 0} \frac{x^2(-\alpha\beta)}{x(1-x)} = \frac{-x^2\alpha\beta}{x - x^2} = 0 = q(0) \checkmark$$

Use the Frobenius method to solve the equation

Indicial equation: $r(r-1)+p(0)r+q(0)$

$$r^2-r+p(0)r+q(0)=0$$

$$r^2-r+\gamma r=0 \qquad r_1=0 \qquad r_2=(1-\gamma)$$

$$r(r-1+\gamma)=0$$

are our characteristic exponents

The solution $y(x)=x^{r_1}\sum_{n=0}^{\infty}a_n x^n=\sum_{n=0}^{\infty}a_n x^n$ is the hypergeometric series.

$$x(1-x)\left(\sum_{n=0}^{\infty}a_n x^n\right)''+(\gamma-(1+\alpha+\beta)x)\left(\sum_{n=0}^{\infty}a_n x^n\right)'-\alpha\beta\sum_{n=0}^{\infty}a_n x^n=0$$

$$y''=\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2} \qquad y'=\sum n a_n x^{n-1}$$

$$=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n \qquad y'=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}-\sum_{n=0}(n+2)(n+1)a_{n+2}x^{n+2}+$$

$$\gamma\sum_{n=0}^{\infty}n a_n x^{n-1}-(1+\alpha+\beta)\sum_{n=0}^{\infty}n a_n x^n-\alpha\beta\sum_{n=0}^{\infty}a_n x^n=0$$

$n'=n+1$ $\qquad$ $n'=n+2$

$$\sum_{n=0}^{\infty}(n+1)n\,a_{n+1}x^n-\sum_{n=0}^{\infty}n(n-1)a_n x^n+$$

$n'=n-1$

$$\sum_{n=0}^{\infty}\gamma(n+1)a_{n+1}x^n-(1+\alpha+\beta)\sum_{n=0}^{\infty}n a_n x^n-\alpha\beta\sum_{n=0}^{\infty}a_n x^n=0$$

Recurrence relation

$$(n+1)n\,a_{n+1}-n(n-1)a_n+\gamma(n+1)a_{n+1}-(1+\alpha+\beta)n\,a_n-\alpha\beta a_n=0$$

$$\left((n+1)n+\gamma(n+1)\right)a_{n+1}-\left[n(n-1)-(1+\alpha+\beta)n-\alpha\beta\right]a_n=0$$

$$a_{n+1}=\frac{\left[n(n-1)-(1+\alpha+\beta)n+\alpha\beta\right]}{(n+1)n+\gamma(n+1)}a_n=\frac{(n+\alpha)(n+\beta)}{(n+1)(n+\gamma)}a_n$$

After multiplication

$$a_n = \frac{(\alpha)_n (\beta)_n}{n!\,(\gamma)_n} a_0 \qquad \text{where} \quad (a)_n = a(a+1)(a+2)\cdots(a+n-1)$$

is rising factorial or Pochammer symbol

Our conclusion $a_0 = 1$

$$y(x) = {}_2F_1(\alpha, \beta \,|\, \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!\,(\gamma)_n} x^n$$

is a solution to the hypergeometric equation.
Called the hypergeometric function

Rmk: If $r_2 = 1-\gamma \notin \mathbb{Z}$, the other solution is

$$x^{1-\gamma}\, {}_2F_1(1+\alpha-\gamma, 1+\beta-\gamma \,|\, 2-\gamma; x)$$

$${}_mF_n(\alpha_1, \alpha_2, \ldots \alpha_m \,|\, \beta_1, \ldots, \beta_n; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_m)_k}{k!\,(\beta_1)_k \cdots (\beta_n)_k} x^k$$

is the generalized hypergeometric series.