ODE 527

Goals.
1) Finish case $r_1 - r_2 \in \mathbb{Z}$ of the Frobenius method
2) Singular points at ∞ : change of variables & why hypergeometric functions show up everywhere
$${}_2F_1(\alpha,\beta|\gamma;x)$$
3) Chebyshev polynomials ⇝ Bessel functions

Recall: The Frobenius method for solving singular ODEs
- nth order similarly

$$P(x)y'' + Q(x)y' + R(x)y = 0$$
$$\Downarrow$$
$$x^2 y'' + x\,p(x)\,y' + q(x)\,y = 0$$

- $p(0)$ well defined, $p(x)$ analytic
- $q(0)$ well defined, $q(x)$ analytic

Frobenius ansatz

If $r(r-1) + r\,p(0) + q(0) = 0$ has root

indicial equation

characteristic exponents – r_1 and r_2

The ansatz

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n \qquad y_2 = x^{r_2}\sum_{n=0}^{\infty} a_n x^n \qquad r_2 - r_1 \notin \mathbb{Z}$$

If $r_1 - r_2$ is an integer: we may not get a second solution because we're basically making the same guess.

$r_1 < r_2$, we can find
$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$$
as a solution

if $k = r_2 - r_1$ then our ansatz would be
$$y_2 = x^{r_2}\sum_{n=0}^{\infty} b_n x^n = x^k x^{r_1}\sum_{n=0}^{\infty} b_n x^n = x^{r_1}\sum_{n=0}^{\infty} b_n x^{n+k}.$$

We have to try something different in this case

Frobenius ansatz if $r_2 - r_1 \in \mathbb{Z}$

If r_1 & r_2 are roots of indicial equation

$r_2 < r_1$ $\qquad r_2 - r_1 \in \mathbb{Z}$

Ansatz

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$

$$y_2 = c\, y_1(x) \cdot \log(x) + x^{r_2} \sum_{n=0}^{\infty} b_n x^n \qquad c \in \mathbb{R}$$

Eg: Euler equation

$$x^2 y'' + \alpha x y' + \beta y = 0 \qquad \alpha, \beta \in \mathbb{R}$$

$$I(r) = r(r-1) + \alpha r + \beta = 0$$

$$r^2 + (\alpha - 1) r + \beta = 0$$

Eg: $\beta = 4 \quad \alpha = -3$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 \qquad r_1 = 2,\ r_2 = 2$$

$r_1 - r_2$ is an integer. We learned that

$y_1 = x^{r_1}$ is a solution $\qquad$ last line: $r_1 - r_2 \notin \mathbb{Z}$ $\left(c_1 x^{r_1} + c_2 x^{r_2} \right)$

Our ansatz is

$$y_2(x) = c\, y_1(x) \log(x) + x^{r_2} \sum_{n=0}^{\infty} b_n x^n .$$

in this case: $b_n = 0$ for $n = 0, 1, \ldots$

Solution is

$$y_2(x) = y_1(x) \log(x) = x^{r_1} \log(x) = x^2 \log(x)$$

$$= \log(x) e^{2\log(x)} \sim t e^{2t} \quad \text{if } t = \log(x)$$

In general: We are solving a Euler equation with varying α, β

So we need the $x^{r_2} \sum_{n=0}^{\infty} b_n x^n$ as well.

Last time: The hypergeometric function

- $${}_2F_1(\alpha,\beta|\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} x^n$$

$(a)_n = a(a+1)\cdots(a+n-1)$ is rising factorial / Pochhammer symbol

- solution around $x=0$ to

$$x(1-x)y'' + (\gamma-(1+\alpha+\beta)x)y' - \alpha\beta y = 0$$

$r_1 = 0 \quad r_2 = 1-\gamma$ if $r_2 - r_1 \notin \mathbb{Z}$ we have other solution

$$x^{1-\gamma}\,{}_2F_1(\alpha',\beta'|\gamma';x)$$

if $r_2 - r_1 \in \mathbb{Z}$

$C\log(x)\,{}_2F_1(\alpha,\beta|\gamma;x)$ + more complicated series.

Claim: The hypergeometric ODE has regular singular points at $0, 1, \infty$

0: What we did before

1: $$x(1-x)y'' + (\gamma-(1+\alpha+\beta)x)y' - \alpha\beta y = 0$$

$$\lim_{x\to 1} (x-1)\frac{Q(x)}{P(x)} = \lim_{x\to 1} \frac{(x-1)\cdot(\gamma-(1+\alpha+\beta)x)}{x(1-x)} = \gamma-1-\alpha-\beta \quad \checkmark$$

$$\lim_{x\to 1} (x-1)^2\frac{R(x)}{P(x)} = \frac{-(x-1)^2\alpha\beta}{x(1-x)} = 0 \quad \checkmark$$

Q: How do we check at ∞?

A: Change of variables $\xi = \frac{1}{x}$ $\quad \lim_{x\to\infty}\frac{1}{x} = 0$ $\quad$ study $x=\infty$ by studying $\xi = 0$

First: Simpler example the ODE for $\sinh(x)$ and $\cosh(x)$

$$y'' = y \quad \rightsquigarrow \quad y'' - y = 0$$

– Looks like it has no singular points (constant coefficients!) but actually at $x=\infty$ it does

$$y''(x) - y(x) = 0 \qquad \xi = \frac{1}{x} \qquad \frac{d}{dx} = \frac{d\xi}{dx}\cdot\frac{d}{d\xi} = \frac{-1}{x^2}\frac{d}{d\xi}$$

$$\frac{d\xi}{dx} = \frac{-1}{x^2}$$

$$\frac{dy}{dx} = \frac{d\xi}{dx}\frac{dy}{d\xi} = \frac{-1}{x^2}\frac{dy}{d\xi}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-1}{x^2}\frac{dy}{d\xi}\right) = \frac{d}{dx}\left(-\xi^2\frac{dy}{d\xi}\right)$$

$$= \left(-\xi^2\frac{d}{d\xi}\right)\left(-\xi^2\frac{dy}{d\xi}\right)$$

$$= -\xi^2\left(-2\xi\frac{dy}{d\xi} - \xi^2\frac{d^2y}{d\xi^2}\right)$$

$$= 2\xi^3\frac{dy}{d\xi} + \xi^4\frac{d^2y}{d\xi^2}$$

$$y''(x) - y(x) = \boxed{2\xi^3 y'(\xi) + \xi^4 y''(\xi) - y(\xi) = 0}$$

For this equation

$\xi = 0$ is an irregular singular point

$$\lim_{\xi\to 0} \frac{\xi Q(\xi)}{P(\xi)} = \lim_{\xi\to 0}\frac{\xi^5}{2\xi^3} = 0 \quad ✓$$

$$\lim_{\xi\to 0} \frac{\xi^2 R(\xi)}{P(\xi)} = \lim_{\xi\to 0}\frac{-\xi^2}{2\xi^3} \longrightarrow -\infty \quad ✗$$

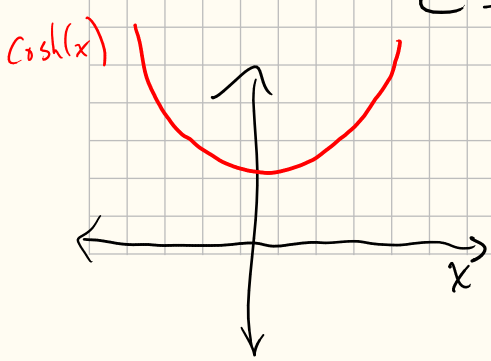
This is an irregular singular point: reflected in the fact that $\cosh(x)$ has really large asymptotics at $x\to\infty$, i.e. $\cosh\left(\frac{1}{\xi}\right)$ has bad asymptotics at $\xi = 0$

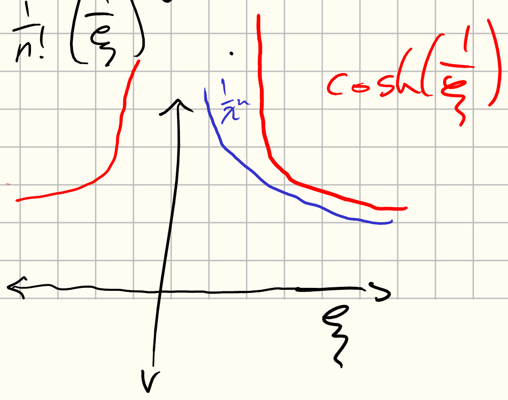
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\cosh\left(\frac{1}{\xi}\right) = \frac{e^{\frac{1}{\xi}} + e^{-1/\xi}}{2} \to \infty \text{ at } \xi = 0 \text{ faster than any polynomial}$$

$$\lim_{\xi\to 0} \xi^N \cosh\left(\frac{1}{\xi}\right) = \infty \text{ for all } N$$

because $e^{\frac{1}{\xi}} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\xi}\right)^n$





We could never find a solution

$$\cosh\left(\frac{1}{\xi}\right) = \xi^{r_1}\sum_{n=0}^{\infty} a_n \xi^n$$ using the Frobenius method.

This behavior doesn't happen for ${}_2F_1(\ldots)$ after changes of variables

Check:
$x=\infty$ is a regular singular point of

$$x(1-x)y'' + (\gamma-(1+\alpha+\beta)x)y' - \alpha\beta y = 0$$

$$\xi = \frac{1}{x} \qquad x = \frac{1}{\xi}$$

$$\frac{dy}{dx} = -\xi^2\frac{dy}{d\xi}$$

$$\frac{d^2y}{dx^2} = \xi^4\frac{d^2y}{d\xi^2} + 2\xi^3\frac{dy}{d\xi}$$

$$\frac{1}{\xi}\left(1-\frac{1}{\xi}\right)\left(\xi^3 y''(\xi) + 2\xi^2 y'(\xi)\right) - \left(\gamma - (1+\alpha+\beta)\frac{1}{\xi}\right)\xi^2 y'(\xi) - \alpha\beta y = 0$$

$$\left(1-\frac{1}{\xi}\right)y'' - \left[\left(1-\frac{1}{\xi}\right)2\xi - \gamma - (1+\alpha+\beta)\frac{1}{\xi}\right)\xi^2\Big]y'(\xi) - \alpha\beta y = 0$$

$\xi = 0$ $$\lim_{\xi\to 0} \frac{\xi Q(\xi)}{P(\xi)} = \frac{\xi\ \xi^2(\text{finite})}{(\text{finite})} \longrightarrow \text{finite} \quad \checkmark$$

$$\lim_{\xi\to 0} \frac{\xi^2 R(\xi)}{P(\xi)} = \frac{\xi^2\alpha\beta}{1-\frac{1}{\xi}} \to 0 \quad \checkmark$$

Conclusion: $0, 1,$ and ∞ are the only singular points & they are regular.

Fact: Any 2nd order linear ODE with 3 regular singular pts (includes ∞) is equivalent to the hypergeometric ODE after a change of variables given by a Möbius transformation

$$x \longmapsto \frac{ax+b}{cx+d} \qquad a,b,c,d \text{ constants}.$$

Remark: Given x_1, x_2, x_3 the transformation

1) $$x \mapsto \frac{(x-x_1)(x_2-x_3)}{(x-x_3)(x_2-x_1)}$$

sends $x_1, x_2, x_3 \longmapsto 0, 1, \infty$

2) Essentially: ∞ being a regular singular point $\Rightarrow$ $P(x)y'' + Q(x)y' + R(x) = 0$ has low-degree polynomials for $P(x), Q(x), R(x)$

Eg: Chebyshev (and other) orthogonal polynomials

$\boxed{\text{see: demo}}$

Chebyshev eq: $$(1-x^2)y'' - xy' + n^2 y = 0$$

Regular singular points at $1, -1, \infty$ ← check

Write the Chebyshev equation as hypergeometric eq. after Möbius transformation

$1 \mapsto 0 \;;\; -1 \mapsto 1 \;;\; \infty \mapsto \infty$

Möbius transformation $x \mapsto \frac{1}{2}(1-x) = \frac{-\frac{1}{2}x + \frac{1}{2}}{0x+1}$
"x'"

$$\frac{d}{dx} = \frac{dx'}{dx}\frac{d}{dx'} = -2\frac{d}{dx'}$$

$$x(1-x)y'' - \left(x - \tfrac{1}{2}\right)y' + n^2 y = 0$$

This is the Hypergeometric eq with $\gamma = \frac{1}{2}$, $\alpha = n$, $\beta = -n$

Therefore ${}_2F_1\left(n, -n \,\middle|\, \tfrac{1}{2}; \tfrac{1}{2}(1-x)\right)$ is a solution to the Chebyshev equation (actually the polynomial solutions)

$$T_n(x) = {}_2F_1\left(n, -n \,\middle|\, \tfrac{1}{2}; \tfrac{1}{2}(1-x)\right)$$

Recall: $T_n(\cos(\theta)) = \cos(n\theta)$

$$\Rightarrow \quad {}_2F_1\left(n, -n \,\middle|\, \tfrac{1}{2}; \tfrac{1}{2}(1-\cos(\theta))\right) = \cos(n\theta)$$

$$\boxed{\sum_{k=0}^{n} \frac{(k)_n(-k)_n}{n!\,(\frac{1}{2})_n}\left(\frac{1}{2}-\frac{1}{2}\cos(\theta)\right)^n = \cos(n\theta)}$$

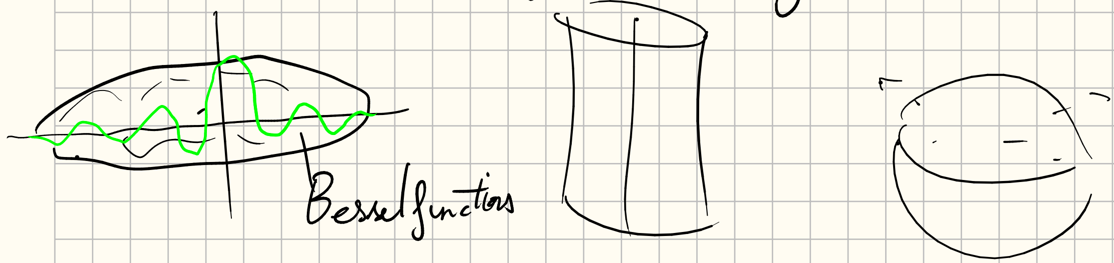
One other application: (one approach to HW#6, Problem 4)

Application to Bessel functions: solutions to

$$x^2y'' + xy' + (x^2-\nu^2)y = 0$$

Bessel equation of order ν.

This arises in wave dynamics in cylindrical/spherical coordinates



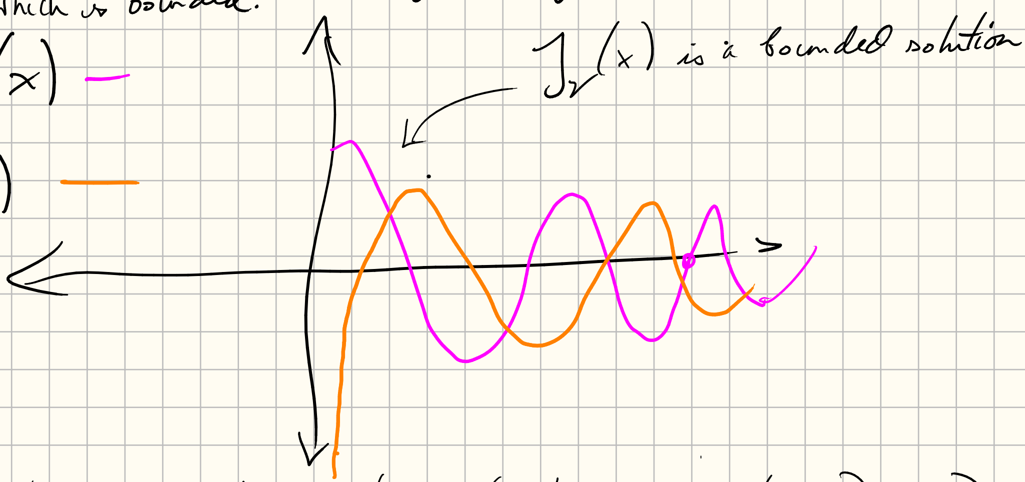
There is one solution around regular regular singular pt $x=0$ which is bounded:

Bessel functions of the first kind

$J_\nu(x)$ —

$Y_\nu(x)$ —

= second kind



The Bessel equation has characteristic exponents $\nu, -\nu$ (assume $\nu \geq 0$)

Always one bounded solution

$$\boxed{J_\nu(x) = x^\nu \cdot \Sigma a_n x^n = \text{Bessel function of the first kind}}$$

Two cases: $r_2 - r_1 \in \mathbb{Z}$

$J_{-\nu}(x)$ is the other solution but we take a different linear combination

$$\boxed{Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} = \text{Bessel function of the second kind}}$$

Other case: $r_1 - r_2 \in \mathbb{Z}$ (i.e. $\nu \in \frac{1}{2}\mathbb{Z}$)

$$\boxed{\begin{aligned} Y_n(x) &= y_2(x) \text{ from Frobenius method} \\ &= c\log(x) J_n(x) + x^{-n}\sum \cdots \\ &\left(= \lim_{\nu \to n} \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \right) \end{aligned}}$$

Other approach:

$J_\nu(x) =$ limit of ${}_2F_1(\alpha, \beta | \gamma; x)$ where the regular points @ 1 & ∞ collide

Bessel eq. has an irregular singular pt @ ∞

Ex:

$$\lim_{\beta \to \infty} {}_2F_1(\alpha, \beta | \gamma; \tfrac{x}{\beta}) = {}_1F_1(\alpha | \gamma; x)$$ – irregular singular pt @ ∞

$$\lim_{\alpha \to \infty} {}_1F_1(\alpha | \gamma; x/\alpha) = {}_0F_1(|\gamma; x)$$

$$\boxed{J_\nu(x) = c\,x^\nu\,{}_0F_1(|\nu+1; -\tfrac{1}{4}x^2)}$$

– This is a great way to write $J_\nu(x)$ as a power series.

(See demo)