ODEs 6-1

Goal: • Asymptotics of Frobenius ansatz solutions (and solutions near singular points)
• Laplace transform

Asymptotics of solutions to singular ODEs

Before:

$$\frac{d^n}{dt^n} y = F(t, y, y', \dots, y^{(n-1)})$$

Only works for non-singular ODEs

$$x^2 y'' + x p(x) y' + q(x) y = 0$$

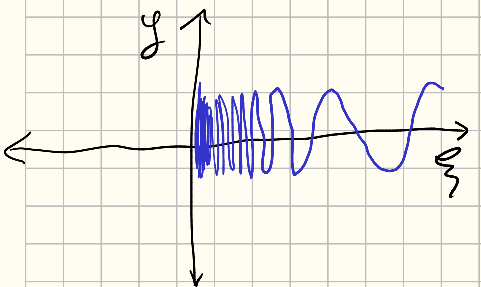
Our theorems don't hold at $x = 0$.

Qualitatively, what happens at $x = 0$?

Irregular singular points

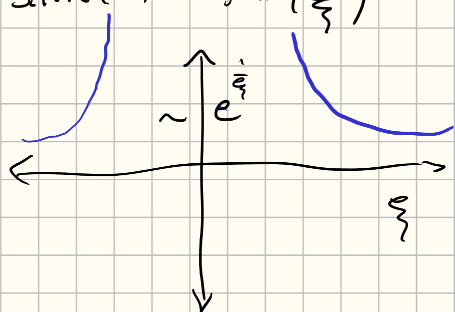
Eg: $y'' + y = 0$ at $x = \infty$

$\sin(x) \sim \sin\left(\frac{1}{\xi}\right)$ $\xi = \frac{1}{x}$



$y'' - y = 0$ at $x = \infty$

$\sinh(x) \sim \sinh\left(\frac{1}{\xi}\right)$



Regular singular points

(non-resonant case)

r_1 and r_2 be characteristic exponents

$r_2 - r_1 \notin \mathbb{Z}$

Solutions: $y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$ and $x^{r_2} \sum_{n=0}^{\infty} b_n x^n = y_2(x)$

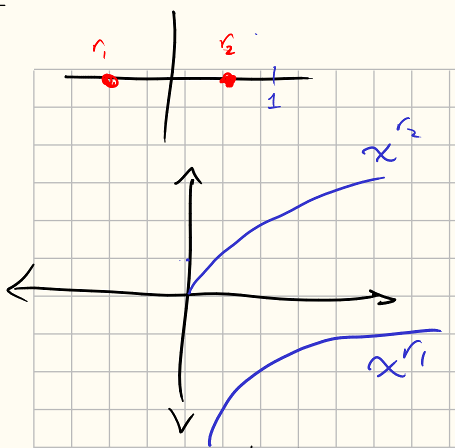
via the Frobenius method

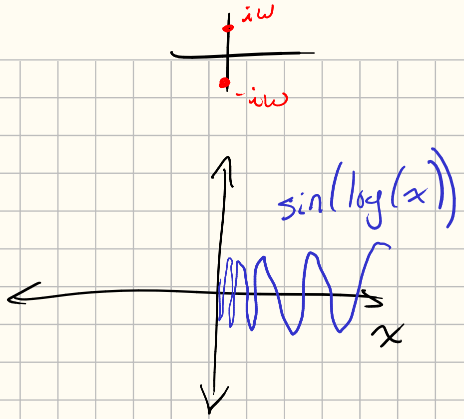
Near $x = 0$ $y_1 \sim x^{r_1}$ (or $x^{r_1 + k}$ $k \in \{0, 1, 2, \dots\}$

$y_2 \sim x^{r_2}$ (or $x^{r_2 + k}$ $k \in \{0, 1, 2, \dots\}$

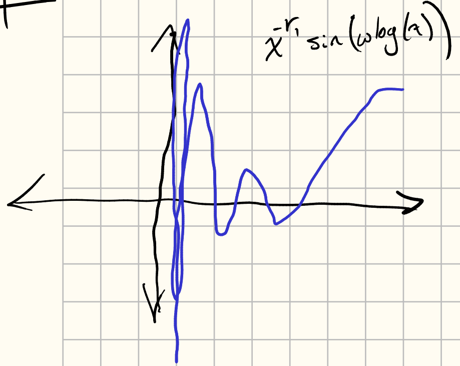
Our qualitative behavior follows almost exactly the same pattern as for linear systems: depends on the location of r_1 & r_2 in the complex plane

Eg:





→ so at $x=0$ but we can still approach the solution using power series



Resonant case: $(r_2 - r_1 \in \mathbb{Z})$

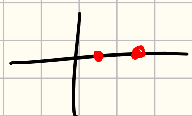
The behavior is less able to be understood from the equation: you need to find a solution at least for the smaller root

$r_2 > r_1$

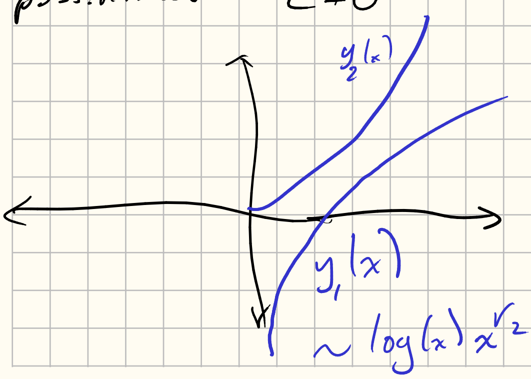
Solutions: $y_2(x) = x^{r_2}\sum_{n=0}^{\infty} a_n x^n$

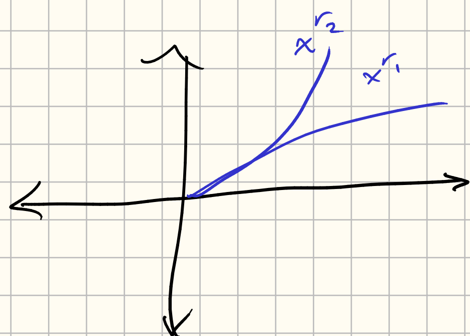
$$y_1(x) = C\log(x)\, y_2(x) + x^{r_1}\sum_{n=0}^{\infty} b_n x^n$$

Eg:



Two possibilities $C \neq 0$ $C = 0$





Eg: Bessel functions near $x=0$

Solutions to
$$x^2y''+xy'+(x^2-\nu^2)y=0$$

Reduces after $z=-\frac{1}{4}x^2$ to

a version of the generalized hypergeometric equation

Eg: $$xy''+\gamma y'-y=0 \qquad (\gamma=\nu+1)$$

Indicial equation

$$r(r-1)+p(0)r+q(0)=0$$

$$x^2y''+\gamma xy'-xy=0 \qquad p(x)=\gamma \quad q(x)=-x$$
$$p(0)=\gamma \quad q(0)=0$$

$$I(r)=r(r-1)+\gamma r=0$$
$$r(r-1+\gamma)=0 \qquad r=0,\ r=-\gamma+1 \qquad \text{assume } \gamma=\nu+1$$
$$r_2=0 \quad r_1=-\gamma+1 \qquad \gamma\geq 0 \Rightarrow$$
$$r_2>r_1$$

$$y(x)=x^{r}\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^n$$

Plug into $xy''+\gamma y'-y=0$

$$y=\sum_{n=0}^{\infty}a_nx^n$$
$$y'=\sum_{n=0}^{\infty}na_nx^{n-1}$$
$$y''=\sum_{n=0}^{\infty}n(n-1)a_nx^{n-2}$$

$$x\left(\sum_{n=0}^{\infty}n(n-1)a_nx^{n-2}\right)+\gamma\left(\sum_{n=0}^{\infty}na_nx^{n-1}\right)-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_nx^{n-1}+\sum_{n=0}^{\infty}n\gamma a_nx^{n-1}-\sum_{n=0}^{\infty}a_nx^n=0$$

$n'=n-1$ $\qquad$ $n'=n-1$

$$\sum_{n'=-1}^{\infty}(n'+1)n'a_{n'+1}x^{n'}+\sum_{n=0}^{\infty}(n'+1)\gamma a_{n'+1}x^{n'}-\sum_{n=0}^{\infty}a_nx^n=0$$

$$(n+1)n\,a_{n+1}+(n+1)\gamma a_{n+1}-a_n=0$$

Solve for a_{n+1}

$$(n+1)(\gamma+n)a_{n+1} = a_n$$

$$a_{n+1} = \frac{a_n}{(n+1)(\gamma+n)}$$

Eg: $a_0 = a_0 \quad a_1 = \frac{a_0}{2\cdot(\gamma+1)} \quad a_2 = \frac{a_1}{(2+1)(\gamma+2)} = \frac{a_0}{3\,2\,(\gamma+1)(\gamma+2)}$

$$a_n = \frac{a_0}{n!\,\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)} = \frac{a_0}{n!\,(\gamma)_n} \qquad (a)_n = a(a+1)(\cdots)(a+n-1)$$

$$y(x) = \sum_{n=0}^{\infty} \frac{a_0}{n!\,(\gamma)_n} x^n = a_0 \sum_{n=0}^{\infty} \frac{1}{n!(\gamma)_n} x^n$$

$$= a_0\, {}_0F_1(\,|\gamma\,;x)$$

After $z = -\frac{1}{4}x^2$

$J_\nu(x)$ one solution to $x^2y'' + xy' + (x^2-\nu^2)y = 0$
is given by $\gamma = \nu+1$

$$= c x^\nu\, {}_0F_1\left(\,|\nu+1\,; -\frac{1}{4}x^2\right).$$

This is the Bessel function of the first kind

Eg: Power series representation

$$J_\nu(x) = c x^\nu\, {}_0F_1(\,|\nu+1\,;z)\Big|_{z=-\frac{1}{4}x^2}$$

$$= c x^\nu \sum_{n=0}^{\infty} \frac{1}{n!(\nu+1)_n}\left(-\frac{1}{4}x^2\right)^n$$

$$= c x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(\nu+1)_n}\,\frac{x^{2n}}{2^{2n}}$$

The other solution? $\quad x^2y'' + xy' + (x^2-\nu^2)y = 0 \qquad p(x) = 1$
$q(x) = x^2-\nu^2$

$$I(r) = r(r-1) + p(0)r + q(0)$$
$$= r(r-1) + r - \nu^2 = 0$$
$$r^2 - \nu^2 = 0 \qquad r_2 = \nu \;, r_1 = -\nu .$$

Non-resonant case: $r_2 - r_1 \notin \mathbb{Z}$. One other solution comes from the fact that our equation is symmetric under $\nu \to -\nu$.

One other solution: $J_{-\nu}(x)$

We actually want a different linear combination of $J_\nu(x)$ and $J_{-\nu}(x)$

Def: The Bessel function of the second kind

$$\boxed{Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}}$$

Resonant case:

$\nu \in \frac{1}{2} + \mathbb{Z}$,
Done $Y_\nu(x)$ is a solution

$n = \nu \in \mathbb{Z}$.

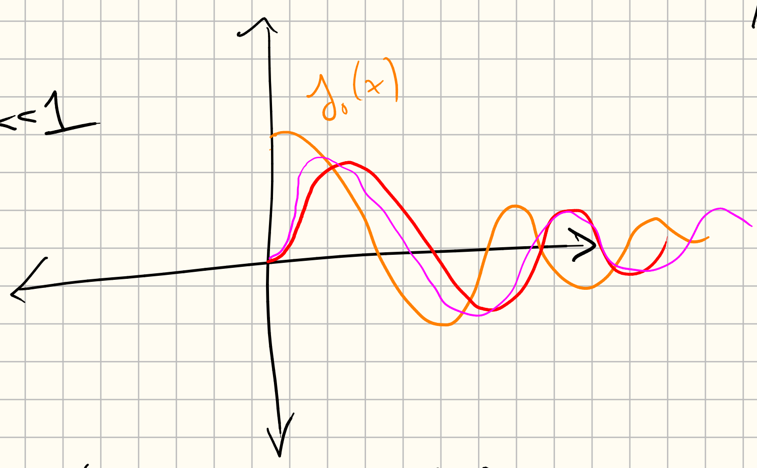
$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$

Frobenius ansatz
$$= c\, J_n(x) \log(x) + x^{-\nu} \sum_{n=0}^{\infty} b_n x^n$$
– b_n are really pretty complicated

Qualitative Behavior?

r_2 root solution

$J_\nu(x)$
$\sim x^\nu \quad x << 1$



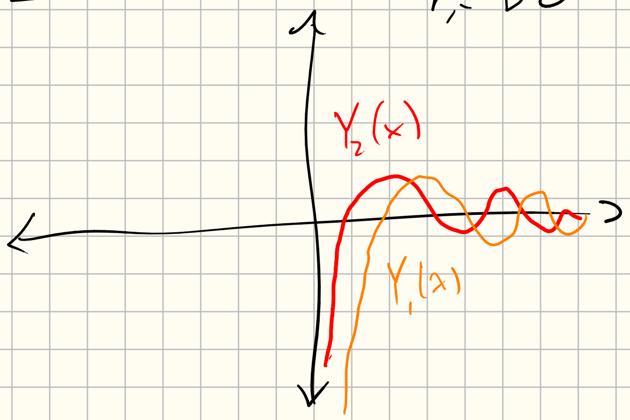
Bessel functions of the first kind

Near ∞ we get $\sim \sqrt{\frac{2}{\pi x}} \cdot \cos(x)$

because our equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$
$$y'' + \frac{y'}{x} + \left(1 - \frac{\nu^2}{x^2}\right) y = 0$$
$$\Big\downarrow x = \infty$$
$$y'' + y = 0$$

Other solutions: (Potentially resonant) $r_1 = -\nu$



$x << 1$

$$Y_\nu(x) \sim x^{-\nu} \quad \nu \notin \mathbb{Z}$$

$$\sim x^{\nu}\log(x) \quad \nu \in \mathbb{Z}$$
$$+ x^{-\nu}$$

By the same argument

$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\cos(x)$$
$$x >> 1.$$

Why do power series methods work?

Answer: The operator $\frac{d}{dx}$ is "Jordan canonical form"

for the "basis" $\left\{ (x-x_0)^0, \frac{(x-x_0)^1}{1!}, \frac{(x-x_0)^2}{2!}, \frac{(x-x_0)^3}{3!}, \dots \right\}$.

Eg: $\frac{d}{dx}(x-x_0)^0 = 0$ $\qquad \frac{d}{dx}\frac{(x-x_0)^n}{n!} = \frac{(x-x_0)^{n-1}}{(n-1)!}$

$\frac{d}{dx}(x-x_0)^1 = (x-x_0)^0$

Write $\frac{d}{dx}$ as a matrix in this "basis"

$\frac{d}{dx}$ is a Linear operator - it is a linear map taking a function to other functions

$$\text{function } f(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \text{function } \frac{d}{dt}f(t)$$

, Linear $\frac{d}{dt}\left(af(t) + bg(t)\right)$
$= a\frac{d}{dt}f(t) + b\frac{d}{dt}g(t)$

In the (Hilbert space) basis $\frac{d}{dx}$ can be written using the matrix

$$\frac{d}{dx} = \begin{array}{c} \\ (x-x_0)^0 \\ (x-x_0)^1 \\ \frac{(x-x_0)^2}{2!} \\ \frac{(x-x_0)^3}{3!} \end{array} \begin{array}{c} \begin{array}{cccc} (x-x_0)^0 & (x-x_0)^1 & \frac{(x-x_0)^2}{2!} & \frac{(x-x_0)^3}{3!} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

– 4x4 Jordan block with $\lambda = 0$

On the vectorspace of approximations $\mathcal{O}(x^{n+1})$ to power series

$\frac{d}{dx}$ is also in J.C.F.

$$\frac{d}{dx} = \begin{array}{cc} & \frac{(x-x_0)^m}{m!} \\ \frac{(x-x_0)^n}{n!} & \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & \Large 0 & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & \Large 0 & & & 0 & 1 \\ & & & & & 0 \end{pmatrix} \end{array}$$

$n \times n$ Jordan block.

Method of Laplace transform

Do the same thing but instead of the basis $\left\{ \frac{(x-x_0)^n}{n!} \right\}$ $n \in \mathbb{Z}$

Use the use the "basis" $\{ e^{st} \}$ $s \in \mathbb{C}$, $t \in \mathbb{R}$.

Differentiation is very nice on e^{st}: $\frac{d}{ds} e^{st} = t\, e^{st}$

Def. If $f(t)$ is a function (the t "coefficient" of e^{st})

The Laplace transform is

$$\boxed{\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt}$$

Sometimes we write $\mathcal{L}\{f(t)\}(s) = F(s)$

Power series methods: Reduce ODEs to systems of equations for coefficients

Laplace transform method: Reduces ODEs to complex algebraic equations

Laplace transform is a transform

$$\underset{f(t)}{\text{function of time}} \longrightarrow \boxed{\mathcal{L}\{f(t)\}(s)} \longrightarrow \underset{F(s)}{\text{function of complex } s \text{ frequency}}$$

Eg: Some Laplace transforms (In practice we build up computations from these)

1) $f(t) = 1$

$$\mathcal{L}[f(t)](s) = \int_0^\infty e^{-st}\,1\,dt = \frac{-1}{s}e^{-st}\Big|_{t=0}^{t=\infty} = 0 - \left(\frac{-1}{s}\right) = \boxed{\frac{1}{s}}.$$

2) $f(t) = e^{at}$

$$\mathcal{L}[e^{at}](s) = \int_0^\infty e^{-st}e^{at}\,dt = \frac{-1}{s-a}e^{-st}\Big|_{0}^{t=\infty} = 0 - \left(\frac{-1}{s-a}\right) = \boxed{\frac{1}{s-a}}.$$