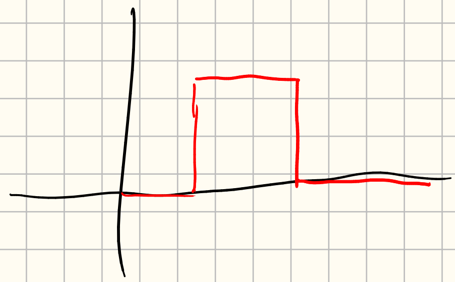
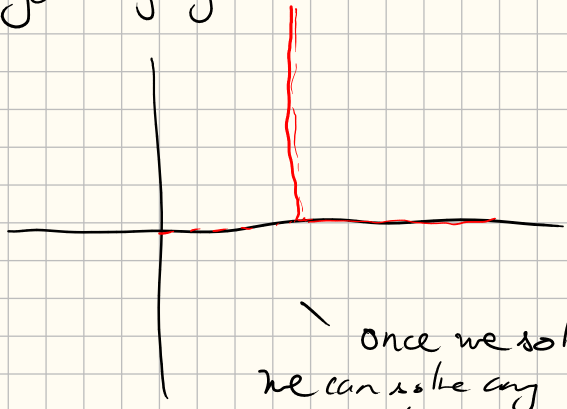
ODEs 6-3

Goal: Laplace transforms for IVPs with discontinuous/singular forcing function

$f(t) =$





once we solve this, we can solve any inhomogeneous problem

How to solve

$$y'' + my' + ky = f(t)$$

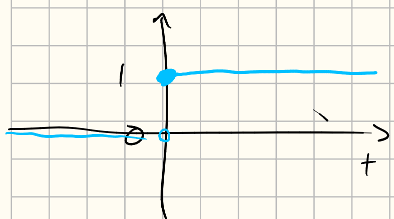
See demo: We want to understand this behavior

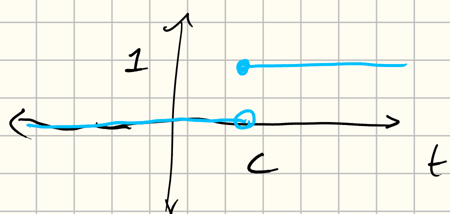
Step functions

Def: The unit step function

1) $$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

2) $$u_c(t) = u(t-c) = \begin{cases} 1 & t \geq c \\ 0 & t < c \end{cases}$$





Remark: We've been implicitly using this the entire time we've studied Laplace transforms.

$$\mathcal{L}\{1\}(s) = \frac{1}{s}$$

What is $\mathcal{L}^{-1}\{\frac{1}{s}\}$? One answer: $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$.

But: • $\mathcal{L}\{u(t)\} = \int_0^\infty e^{-st} u(t)\,dt = \frac{1}{s}$ so

$\mathcal{L}^{-1}\{\frac{1}{s}\} = u(t) \ldots$

• Let $g(t) =$

$\mathcal{L}\{-\}$ only sees this

then

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt = \int_0^\infty e^{-st}\,dt = \frac{1}{s}$$

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt$$

$$= \int_0^c 0\,dt + \int_{c}^\infty e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_c^\infty = \frac{e^{-sc}}{s}.$$

More generally:

• Time shift formula

$$\boxed{\mathcal{L}\{f(t-c)u_c(t)\} = e^{-cs}\,\mathcal{L}\{f(t)\}}$$

PS: $\mathcal{L}\{f(t-c)u_c(t)\} = \int_0^\infty e^{-st} f(t-c) u_c(t)\,dt$

$$= \int_c^\infty e^{-st} f(t-c)\,dt = \int_0^\infty e^{-s(\tau+c)} f(\tau)\,d\tau$$

$\tau = t - c$

$$= e^{-sc}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau = e^{-sc}\mathcal{L}\{f(t)\}$$

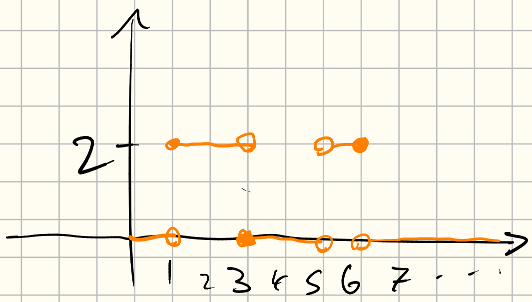
Compare w/ frequency shift formula

$$\mathcal{L}\{e^{at} f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a).$$

Discontinuous forcing functions

We can build discontinuous forcing functions from $u_c(t)$.

Eg:



$$= f(t) = \begin{cases} 2 & \text{if } 1 \le t < 3 \text{ OR } 5 \le t < 6 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$f(t) = 2u_1(t) - 2u_3(t) + 2u_5(t) - 2u_6(t).$$

Therefore

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{u_1(t) - u_3(t) + u_5(t) - u_6(t)\}$$

$$= 2\frac{e^{-s}}{s} - 2\frac{e^{-3s}}{s} + 2\frac{e^{-5s}}{s} - 2\frac{e^{-6s}}{s}.$$

Rmk: We know how to take $\mathcal{L}^{-1}\{P(s)\}$ where

$$P(s) = \frac{b_n s^{n-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0} \quad \text{OR} \quad e^{-cs}\frac{b_n s^{n-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

B/c we can use partial fraction decomposition

$$P(s) = \frac{a_1 s + b_1}{(s-c_1)^2 + d_1} + \cdots + \frac{a_k}{s - b_k}$$

$$\mathcal{L}^{-1}\{P(s)\} = e^{r_1 t}\sin(\omega_1 t) + \cdots + e^{b_k t} = p(t)$$

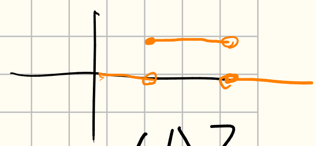
$$\mathcal{L}^{-1}\{e^{-cs}P(s)\} = p(t-c)\,u_c(t).$$

Now we can actually calculate examples

Eg: Solve the IVP

$$y' + y = g(t) \qquad g(t) = \begin{cases} 1 & 5 \le t < 20 \\ 0 & \text{otherwise} \end{cases}$$

$$y(0) = 0$$



Take $\mathcal{L}\{-\}$

$$\mathcal{L}\{y' + y\} = \mathcal{L}\{g(t)\} = \mathcal{L}\{u_5(t) - u_{20}(t)\}$$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\}$$

$$sY(s) - y(0) + Y(s) = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$Y(s) = \frac{1}{s+1}\left(\frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}\right)$$

$$= e^{-5s}\frac{1}{s(s+1)} - e^{-20s}\frac{1}{s(s+1)}$$

Partial fractions $\quad \frac{1}{s(s+1)} = \frac{1}{s} + \frac{-1}{s+1}$

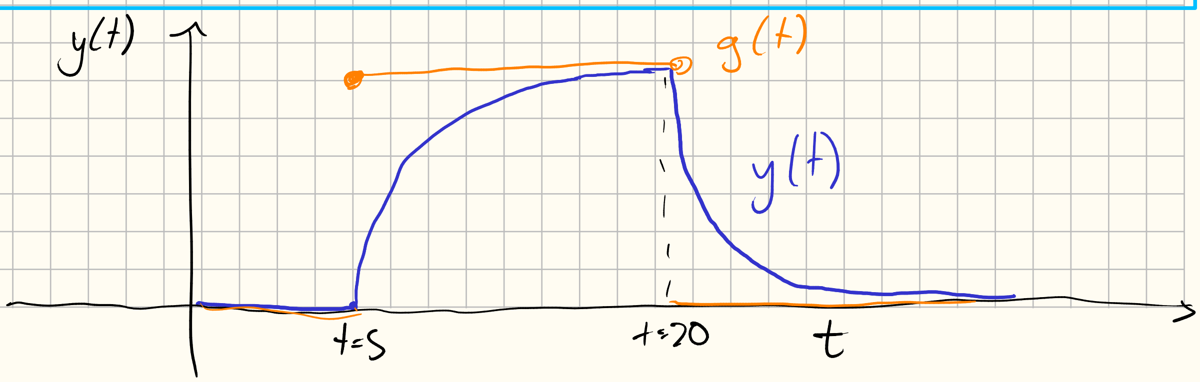
$$Y(s) = e^{-5s}\left(\frac{1}{s} - \frac{1}{s+1}\right) - e^{-20s}\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = u_5(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}(t-5)$$
$$- u_{20}(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}(t-20)$$

Use $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$, $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

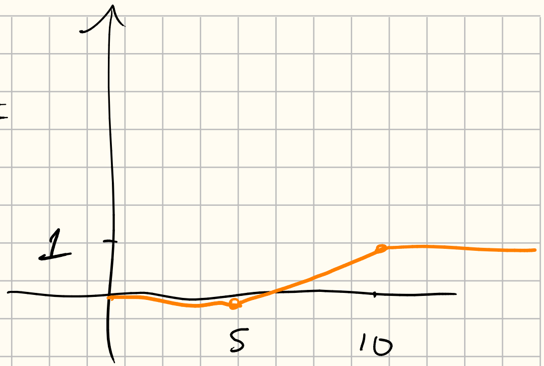
$$\boxed{y(t) = u_5(t)\left(1 - e^{-(t-5)}\right) - u_{20}(t)\left(1 - e^{-(t-20)}\right)}$$

plot:



Eg: (Ramp-forced undamped oscilator)

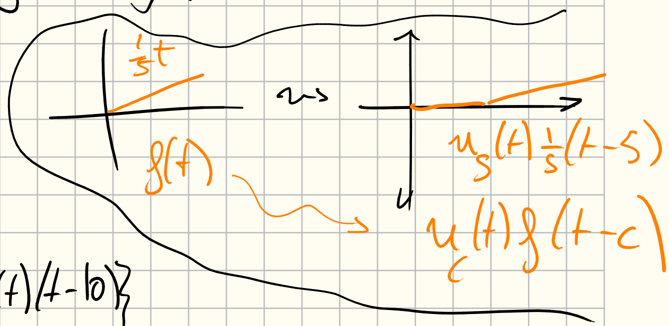
$$y''+4y=g(t) \qquad g(t)=$$



$$y(0)=0 \quad y'(0)=0$$

- Rewrite $g(t)$ using step functions: By building from $t=0,5,10,\ldots$

$$g(t)=\frac{1}{5}\left(u_5(t)(t-5)-u_{10}(t)(t-10)\right)$$



- Take $\mathcal{L}\{-\}$ of both sides

$$\mathcal{L}\{y''\}+4\mathcal{L}\{y\}=\frac{1}{5}\mathcal{L}\{u_5(t)(t-5)\}-\frac{1}{5}\mathcal{L}\{u_{10}(t)(t-10)\}$$

$$s^2Y(s)-sy(0)-y'(0)+4Y(s)=\frac{e^{-5s}}{5}\mathcal{L}\{t\}-\frac{e^{-10s}}{5}\mathcal{L}\{t\}$$

$$(s^2+4)Y(s)=\frac{e^{-5s}}{5}\left(\frac{1}{s^2}\right)-\frac{e^{-10s}}{5}\left(\frac{1}{s^2}\right)$$

$$Y(s)=\frac{1}{5}\left[e^{-5s}\left(\frac{1}{s^2(s^2+4)}\right)-e^{-10s}\left(\frac{1}{s^2(s^2+4)}\right)\right]$$

- Partial fractions

$$\frac{1}{s^2(s^2+4)}=\frac{\frac{1}{4}}{s^2}+\frac{-\frac{1}{4}}{s^2+4}\quad\left(=\frac{\frac{s^2}{4}+1-\frac{s^2}{4}}{s^2(s^2+4)}\right) \quad ✓$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{5}u_5(t)\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s^2}-\frac{1}{4}\frac{1}{s^2+4}\right\}(t-5)$$

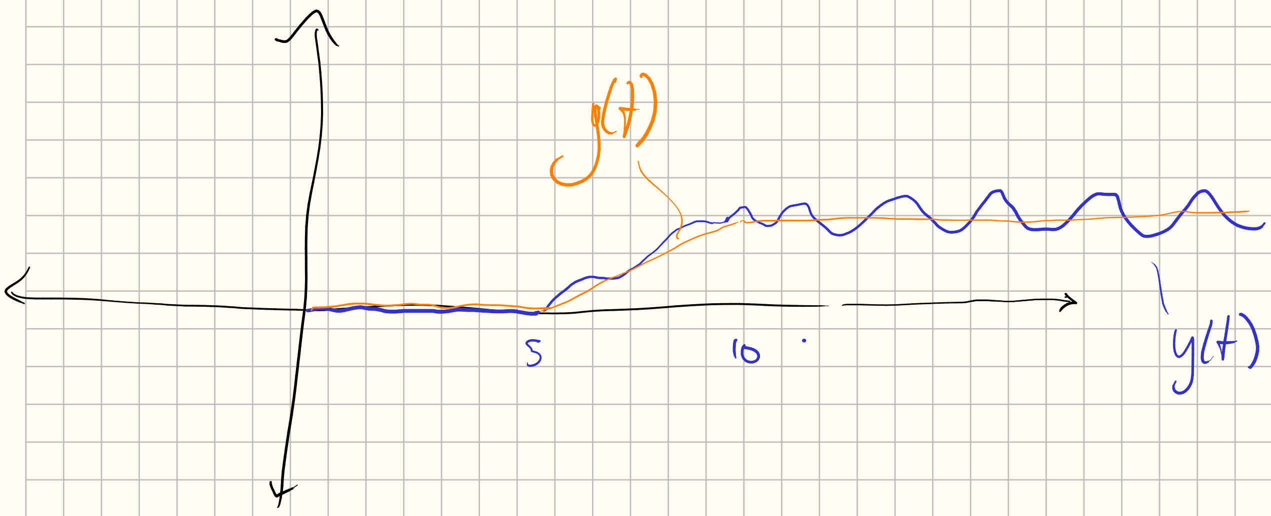
$$-\frac{1}{5}u_{10}(t)\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s^2}-\frac{1}{4}\frac{1}{s^2+4}\right\}(t-10)$$

$\mathcal{L}\{\sin(at)\}=\frac{a}{s^2+a^2}$

$$\boxed{y(t)=\frac{1}{20}\left[u_5(t)\left(t-5-\frac{\sin(2(t-5))}{2}\right)-u_{10}(t)\left(t-10-\frac{\sin(2(t-10))}{2}\right)\right]}$$

Rewrite

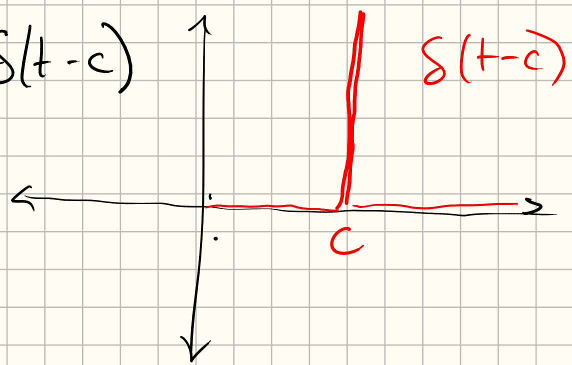
$$y(t) = \frac{1}{4}\left(\frac{1}{5}\left(u_5(t)(t-5) - u_{10}(t)(t-10)\right) - \frac{1}{40}\left(u_5(t)\sin(2t-10) - u_{10}(t)\sin(2t-20)\right)\right)$$



How do we approach $g(t) = \delta(t-c)$

Answer: $\delta(t-c)$ isn't a function

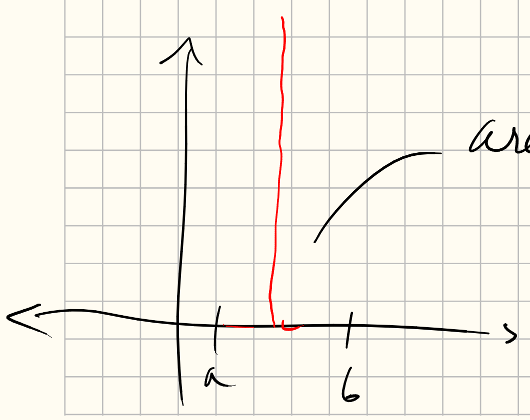
it's a distribution or a generalized function



Def (engineering) - a distribution is like a function, and we can integrate it according to some rules

Eg:

- $$\int_a^b \delta(t)\,dt = \begin{cases} 1 & \text{if } 0 \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$
- $$\int_a^b g(t)\,\delta(t-c)\,dt = \begin{cases} g(c) & \text{if } c \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$



area under = 1

$$= \int_a^b \delta(t-c)\,dt$$

Def: (Mathematics definition)

A distribution D is a particular type of continuous linear map

from $D : \{\text{Test functions}\} \longrightarrow \mathbb{R}$

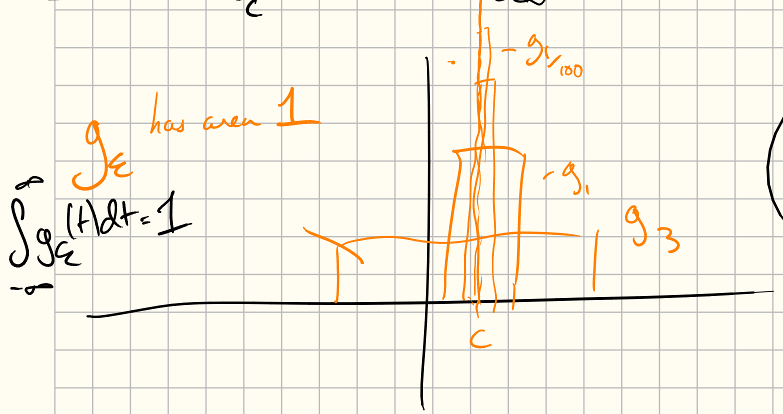
Rmk: • "continuous" has a technical definition but it implies that

$$D(f) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} g_\varepsilon(t) f(t)\,dt$$

(for some sequence of functions $g_\varepsilon(t)$)

Eg:

$$D_{\delta_c}(f) = \int_{-\infty}^{\infty} \delta(t-c) f(t)\,dt = f(c)$$



$$\left(= \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} g_\varepsilon(t) f(t)\,dt = f(c) \right)$$

• Linear b/c

$$D_{\delta_c}(af(t)+bg(t)) = \int_{-\infty}^{\infty} \delta(t-c)(af(t)+bg(t))\,dt$$

$$= a D_{\delta_c}(f(t)) + b D_{\delta_c}(g(t))$$

$$= af(c) + bg(c).$$

We can do lots more

Next time: $\frac{d}{dt}$ for distributions

Eg:

$$\boxed{\frac{d}{dt} u_c(t) = \delta(t-c)}.$$