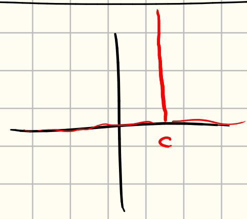
ODEs 6-7

- Multiplication and how it relates distributions and to the Laplace transform

Goal: Formula (using convolution) for the solution to $L[y] = g(t)$ based on the impulse response, (solution to $L[y] = \delta(t)$)

Recall: A distribution (like $\delta(t-c)$) is really a special type of linear operator



$$D : \{\text{Test functions}\} \to \mathbb{R}$$

like $D_{\delta_c} = \int_{-\infty}^{\infty} \delta(t-c) f(t)\,dt = \left[f \longmapsto f(c) \right]$

§ Operations on distributions

1) Addition

$$(D_1 + D_2)[f] = D_1[f] + D_2[f]$$

Test function

Eg: $\int_{-\infty}^{\infty} \left(\delta(t-a) + \delta(t-b)\right) f(t)\,dt = f(a) + f(b)$

2) Multiplication?

(Distribution) × (function) → (Distribution) ↙ This is okay

Eg: $\delta(t-c) \times g(t) \longrightarrow \delta(t-c)g(t)$

$$D_{\delta_c} \cdot g(t)[f] = \int_{-\infty}^{\infty} \delta(t-c)g(t)\, f(t)\,dt = g(c)f(c)$$

More generally:

D any distribution

$D \cdot g(t)\,[f] = D[f \cdot g]$ - Well defined.

What about

(Distribution) $\times$ (Distribution) $\to$ (Distribution) ?

$\boxed{\text{This does not work!}}$

Eg: $$\int_{-\infty}^{\infty} \cancel{\delta(t-c)\cdot\delta(t-c)}\, g(t)\,dt = \cancel{g(c)}^2 \; ?$$

This would be bad

$D_1 : \{\text{test functions}\} \to \mathbb{R}$ & $D_2 : \{\text{test functions}\} \to \mathbb{R}$

3) Differentiation first an example

Eg: Calculate

$$\frac{d}{dt} u_c(t) = \,?$$

Define this using integration by parts (assume $\lim_{t\to\pm\infty} g(t) = 0$)

define as a distribution

$g(t)$

$$\int_{-\infty}^{\infty} \frac{d}{dt}u_c(t)\, g(t)\,dt = u_c(t)g(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_c(t)\cdot\frac{d}{dt}g(t)\,dt$$

$$= -\int_{-\infty}^{\infty} u_c(t)\frac{d}{dt}g(t)\,dt = -\int_{c}^{\infty}\frac{d}{dt}g(t)\,dt$$

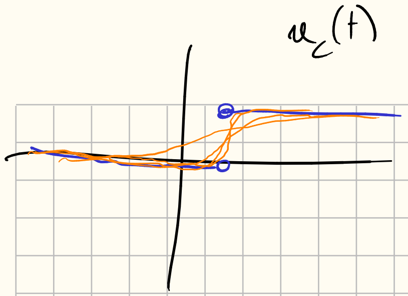
$$= -g(t)\Big|_{c}^{\infty} = g(c).$$

As a distribution $\frac{d}{dt}u_c(t)$ is

$$g(t) \longmapsto \int_{-\infty}^{\infty} \frac{d}{dt}u_c(t) g(t)\,dt = g(c)$$

$$D_{\frac{d}{dt}u_c(t)}[g] \longmapsto g(c)$$

i.e. $D_{\frac{d}{dt}u_c(t)}[g] = D_{\delta}(g)$ OK: $\boxed{\frac{d}{dt}u_c(t) = \delta(t-c)}$



<u>More generally:</u>

The derivative of any distribution

$$D_\varphi[g] = \int_{-\infty}^{\infty} \varphi(t)\, g(t)\, dt \quad \text{can be defined by}$$

$$D_{\frac{d\varphi}{dt}}[g] = D_\varphi\left[-\frac{d}{dt}g\right] \quad \text{(just integration by parts).}$$

§ Impulse response

If we have a linear differential operator $L[y]$

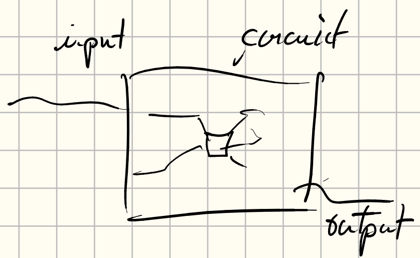
<u>Eg:</u> $L[y] = ay'' + by' + cy$

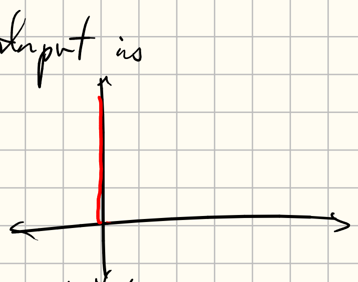
<u>Def:</u> then the function $h(t)$ which solves the IVP

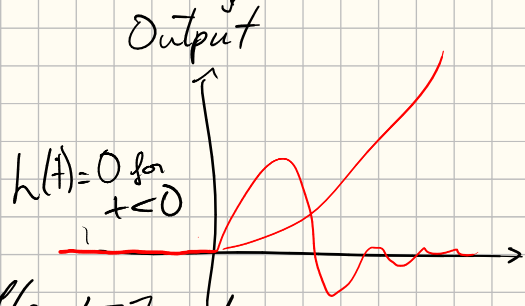
$$L[y] = \delta(t) \qquad y(0)=0,\ y'(0)=0,\ \text{etc.}$$

is called the impulse response of the ODE

Eg:







b/c $L[y]$ is linear.

We can recover the solution to $L[y] = g(t)$ from $h(t)$.

Multiplication and Laplace transforms

If $\mathcal{L}\{f(t)\} = F(s)$
$\mathcal{L}\{g(t)\} = G(s)$

Then what is

$$\mathcal{L}^{-1}\{F(s)G(s)\} \cancel{= f(t)g(t)} \quad ?$$

This is especially useful since

$$\boxed{\begin{aligned} \mathcal{L}\{\delta(t-c)\} &= \int_0^\infty e^{-st}\delta(t-c)\,dt = e^{-sc} \\ \mathcal{L}\{\delta(t)\} &= \lim_{c\to 0}\left(e^{-sc}\right) = 1 \end{aligned}}$$

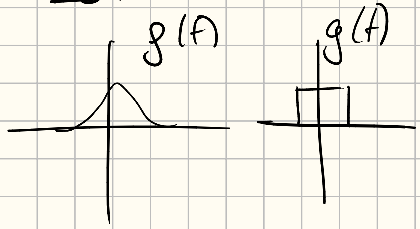
Therefore

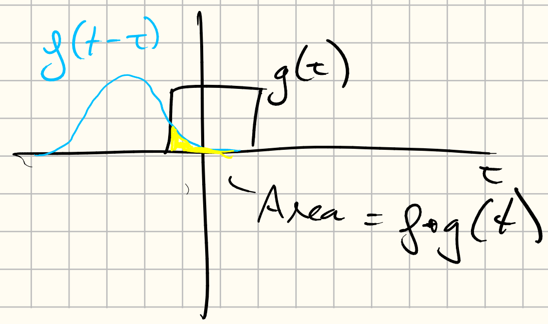
$$\mathcal{L}\{\delta(t)\}\cdot\mathcal{L}\{f(t)\} = \mathcal{L}\{f(t)\}$$

Def: The convolution product of $f(t)$ and $g(t)$ is

$$\boxed{f*g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau}$$

Eg:





It satisfies some nice properties:

Eg: $f*g = g*f$

$f*(ag_1(t)+bg_2(t)) = af*g_1 + bf*g_2$

etc.

Slogan "Convolution is multiplication in the frequency domain"

Thm (Convolution theorem)

If $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{g(t)\} = G(s)$,

$H(s) = F(s)\cdot G(s)$,

$h(t) = \mathcal{L}^{-1}\{H(s)\}$ then

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = f*g(t).$$

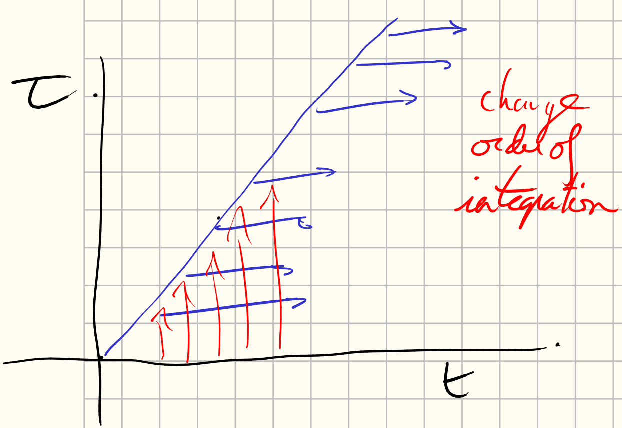
Pf: $$F(s)G(s) = \int_0^\infty e^{-s\xi} f(\xi)\,d\xi \cdot \int_0^\infty e^{-s\tau} g(\tau)\,d\tau$$

$$= \int_0^\infty e^{-s\tau} g(\tau) \int_0^\infty e^{-s\xi} f(\xi)\,d\xi\,d\tau$$

$$= \int_0^\infty g(\tau) \int_0^\infty e^{-s(\tau+\xi)} f(\xi)\,d\xi\,d\tau$$

$t = \xi + \tau$
$d\xi = dt$

$$= \int_0^\infty g(\tau) \int_\tau^\infty e^{-st} f(t-\tau)\,dt\,d\tau$$



$$= \int_0^\infty \int_0^t e^{-st} g(\tau) f(t-\tau)\,d\tau\,dt$$

$$= \int_0^\infty e^{-st} f*g(t)\,dt \qquad \square$$

Eg: Calculate the impulse response of

$L[y] = y'' + 2y' + 2y$, i.e. solve the IVP

$$y'' + 2y' + 2y = \delta(t) \quad , \quad y(0) = y'(0) = 0$$

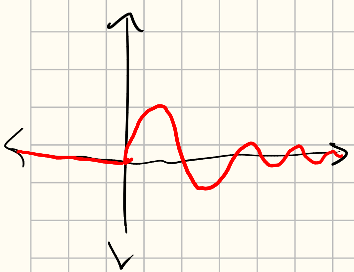
$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t)\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 2Y(s) = 1$$

$$(s^2 + 2s + 2)\, Y(s) = 1$$

$$Y(s) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1}$$

$$\boxed{y(t) = e^{-t} \sin(t) \cdot u_0(t)}$$



Now we can solve $L[y] = g(t)$.

Eg: $y'' + 2y' + 2y = g(t) \qquad y(0) = y'(0) = 0$

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{g(t)\} \qquad G(s) = \mathcal{L}\{g(t)\}$$

$$(s^2 + 2s + 2)\, Y(s) = G(s)$$

$$Y(s) = \frac{1}{s^2 + 2s + 2} G(s)$$

If $G(s)$ were a nice function, we might be able to do partial fraction decomp. & use a table to solve for $y(t)$.

But the RHS is multiplication in the frequency domain: use the convolution theorem:

$$Y(s) = \frac{1}{s^2+2s+2} \cdot G(s)$$

$$= \mathcal{L}\{\text{impulse response}\}\,\mathcal{L}\{g(t)\}$$

So

$$y(t) = h * g\,(t) = \int_0^t h(t-\tau)\,g(\tau)\,d\tau \,.$$

since $h(t) = e^{-t}\sin(t)$,

$$\boxed{y(t) = \int_0^t e^{-(t-\tau)} \sin(t-\tau)\, g(\tau)\, d\tau}$$

We didn't use anything about $g(t)$ (other than that $G(s)$ exists, i.e. $g(t) \in O(e^{at})$).

More generally:

If $L\{y\} = a_n y^{(n)} + \dots + a_0 y$ is a general linear constant coefficient differential operator

then impulse response $h(t)$ solves

$$L\{y\} = \delta(t) \qquad y(0) = y'(0) = \dots y^{(n-1)}(0) = 0$$

$$a_n y^{(n)} + \dots + a_0 y = g(t) \qquad \text{becomes}$$

$$\mathcal{L}\{a_n y^{(n)} + \dots + a_0 y\} = G(s)$$

$$(a_n s^n + \dots + a_1 s + a_0)\, Y(s) = G(s)$$

$$Y(s) = \frac{1}{a_n s^n + \cdots + a_0} G(s) \quad \left(= \frac{1}{a_n s^n + \cdots + a_0} \right.$$ if $g(t) = \delta(t)$, $G(s) = 1$

$$= \mathcal{L}\{\text{impulse response}\} \mathcal{L}\{g(t)\}$$

Therefore

$$y(t) = \int_0^t h(t-\tau) g(\tau) \, d\tau$$. solves

$L[y] = g(t)$ $y(0) = \cdots y^{(n-1)}(0) = 0$.

Remarks:

1) Sometimes the integrals are difficult to solve

2) We can solve more general IVPs

$L[y] = g(t)$ $y(0) = y_0$ $y'(0) = y_0'$, $\ldots$ $y^{(n-1)}(0) = y_0^{(n-1)}$

by finding $y_h(t)$ solving

$L[y] = 0$ $y(0) = y_0$, etc. then the solution is

$$y(t) = y_h(t) + \int_0^t h(t-\tau) g(\tau) \, d\tau$$ since

$y^{(k)}(0) = y_0^{(k)} + 0$ and $L[y(t)] = 0 + L\left[\int_0^t h(t-\tau) g(\tau) \, d\tau\right]$
$= g(t)$.

3) This is a second "general method" of solving inhomogeneous linear ODEs. Whether to use this or variation of parameters depends on the situation.

4) The convolution with the impulse response is like a

inverse matrix to the differential operator.

Observe: We solved

$$\mathcal{L}[y] = g(t) \qquad \text{assuming } y(0) = \cdots y^{(n-1)}(0) = 0$$

by writing

$$y(t) = \int_0^t h(t-\tau) g(\tau) d\tau \qquad \text{so if}$$

$$\mathcal{L}^{-1}[g] := \int_0^t h(t-\tau) g(\tau) d\tau \quad \text{then}$$

$$\mathcal{L}[\mathcal{L}^{-1}[g(t)]] = g(t).$$