ODEs 6-8

Goal: • Linear operators (understand the relationship between ODEs and linear algebra)
• Boundary value problems

There are deep relations between linear algebra and ODEs.

Eg: Solve the IVP

$$\vec{y}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\vec{y} + \begin{pmatrix} \delta(t-3) \\ 2\delta(t-3) \end{pmatrix}; \quad \vec{y}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

using the Laplace transform

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} \delta(t-3) \\ 2\delta(t-3) \end{pmatrix}; \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

uses $\mathcal{L}\{M\vec{y}(t)\} = M\mathcal{L}\{\vec{y}\}$ i.e. linearity

$$\begin{pmatrix} \mathcal{L}\{y_1(t)'\} \\ \mathcal{L}\{y_2(t)'\} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} \mathcal{L}\{y_1(t)\} \\ \mathcal{L}\{y_2(t)\} \end{pmatrix} + \begin{pmatrix} \mathcal{L}\{\delta(t-3)\} \\ 2\mathcal{L}\{\delta(t-3)\} \end{pmatrix}$$

$$\begin{pmatrix} sY_1(s) - y_1(0) \\ sY_2(s) - y_2(0) \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} + \begin{pmatrix} e^{-3s} \\ 2e^{-3s} \end{pmatrix}$$

This is now just an algebraic equation.

Set $\vec{Y}(s) = \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix}$ our equation is

$$s\vec{Y}(s) - \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\vec{Y}(s) + \begin{pmatrix} e^{-3s} \\ 2e^{-3s} \end{pmatrix}$$

$$s = sI = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$

$$\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\right]\vec{Y}(s) = \begin{pmatrix} e^{-3s} + y_1(0) \\ 2e^{-3s} + y_2(0) \end{pmatrix}$$

$y_1(0) = 2 \quad y_2(0) = 2$

$$\begin{pmatrix} s-1 & 2 \\ -3 & s+4 \end{pmatrix} \vec{Y}(s) = \begin{pmatrix} e^{-3s}+2 \\ 2e^{-3s}+2 \end{pmatrix}$$

$$\vec{Y}(s) = \frac{1}{(s-1)(s+4)+6} \begin{pmatrix} s+4 & 2 \\ -3 & s-1 \end{pmatrix} \begin{pmatrix} e^{-3s}+2 \\ 2e^{-3s}+2 \end{pmatrix}$$

$$\vec{Y}(s) = \frac{1}{s^2+3s+2} \begin{pmatrix} (s+4)(e^{-3s}+2) + 2(2e^{-3s}+2) \\ -3(e^{-3s}+2) + (s-1)(e^{-3s}+2) \end{pmatrix}$$

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s^2+3s+2}\left[e^{-3s}(s+6) + 2s+12\right] \\ \frac{1}{s^2+3s+2}\left[e^{-3s}(s-4) + 2s-8\right] \end{pmatrix}$$

So $\frac{1}{s^2+3s+2} = \frac{1}{(s+2)(s+1)} = \frac{1}{s+1} - \frac{1}{s+2}$ $\quad \mathcal{L}^{-1}\{-\} = e^{-t} - e^{-2t}$

$$y_1(t) = \mathcal{L}^{-1}\{Y_1(s)\} = (e^{-t} - e^{-2t}) * \mathcal{L}^{-1}\{e^{-3s}(s+6) + 2s+12\}$$

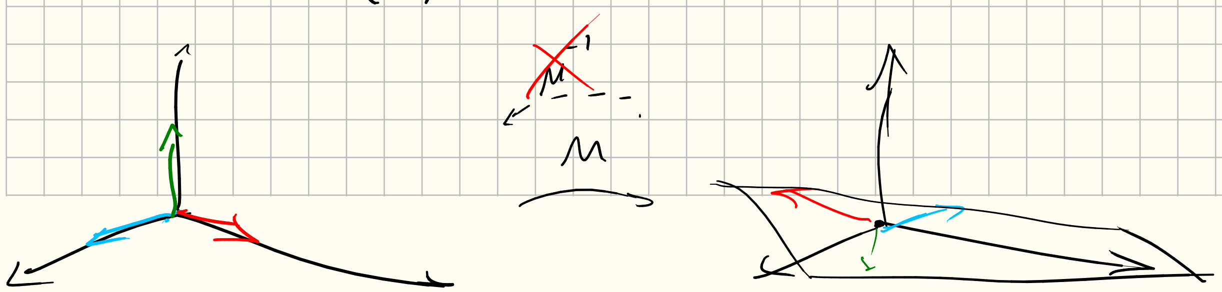
$$y_2(t) = \mathcal{L}^{-1}\{Y_2(s)\} = (e^{-t} - e^{-2t}) * \mathcal{L}^{-1}\{e^{-3s}(s-4) + 2s-8\}.$$

<u>Leading question</u>: How do we find the inverse of the matrix

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix} ?$$

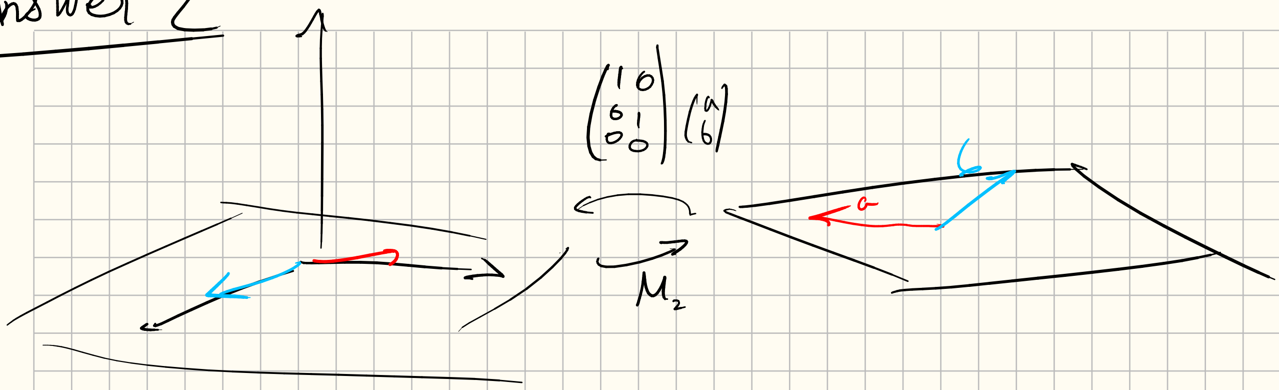
<u>Answer 1</u>: $\det(M) = 0$ so we can't.

(e.g. $M\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$, $\vec{c}_1 + \vec{c}_2 = \vec{c}_3$).



Answer 2

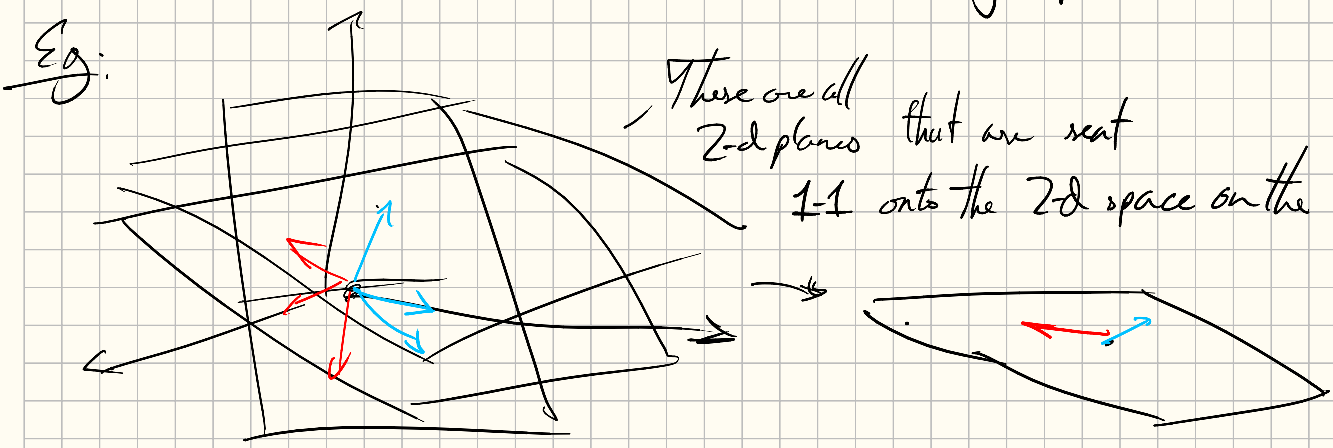
Sure we can: $a \mapsto \begin{pmatrix}1\\0\\0\end{pmatrix}$ $b \mapsto \begin{pmatrix}0\\1\\0\end{pmatrix}$



The biggest problem is that the answer isn't unique:
Given $\vec{v}$ on the LHS, $\vec{v} \neq \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix} M_2 \vec{v}$

unless $\vec{v}$ lies in the xy-plane

Eg:



These are all 2-d planes that are sent 1-1 onto the 2-d space on the RHS.

Answer 3:

The inverse of M applied to $\begin{pmatrix}a\\b\end{pmatrix}$ isn't a unique vector, but it's all solutions to

$$M\vec{v} = \begin{pmatrix}a\\b\end{pmatrix} \qquad \text{so}$$

$$\text{“}M^{-1}\text{”}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix} \quad \text{since} \quad M\begin{pmatrix}1\\1\\-1\end{pmatrix} = \vec{0}.$$

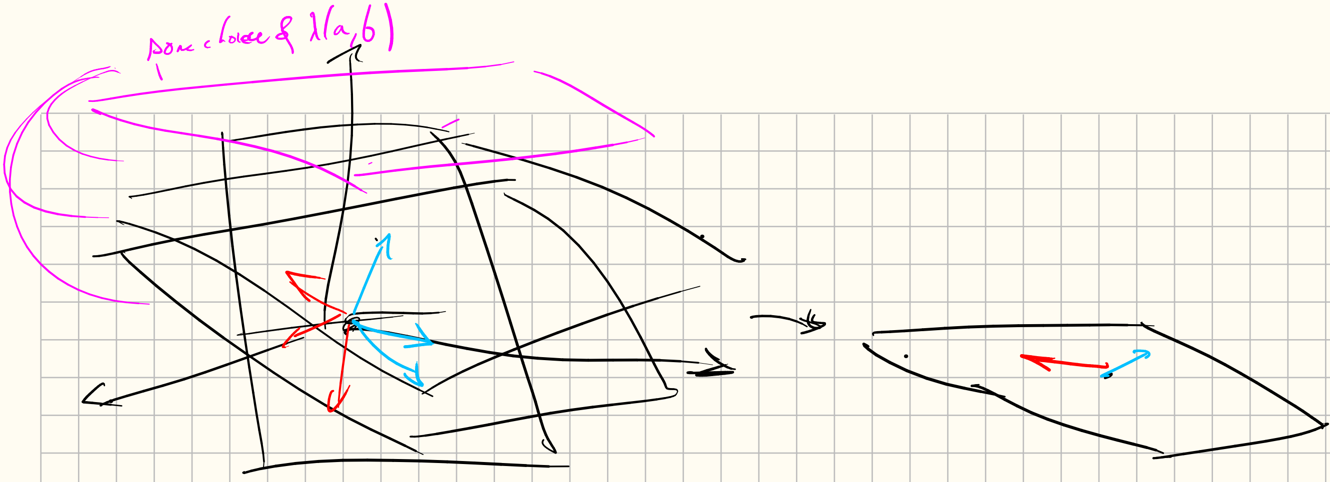
$$M\,\text{“}M^{-1}\text{”}\begin{pmatrix}a\\b\end{pmatrix} = M\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} + \lambda M\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}a\\b\end{pmatrix} + 0 = \begin{pmatrix}a\\b\end{pmatrix}.$$

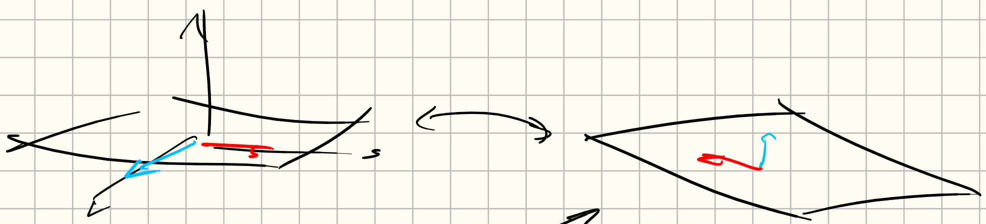
Answer 4

For every $\begin{pmatrix}a\\b\end{pmatrix}$, we pick a specific λ from the last problem and M is invertible when we restrict to vectors of the form

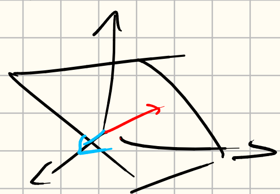
$$\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} + \lambda(a,b)\begin{pmatrix}1\\1\\-1\end{pmatrix}.$$

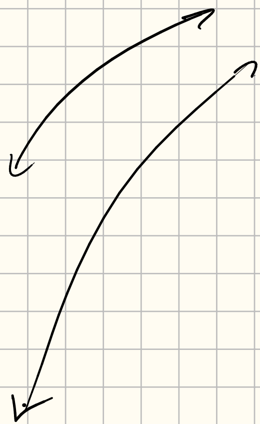


Eg: $\lambda(a,b)=0$

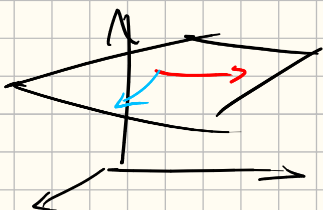


$\lambda(a,b)=a$





$\lambda(a,b)=1$
(not linear)



This isn't unrelated to ODEs:

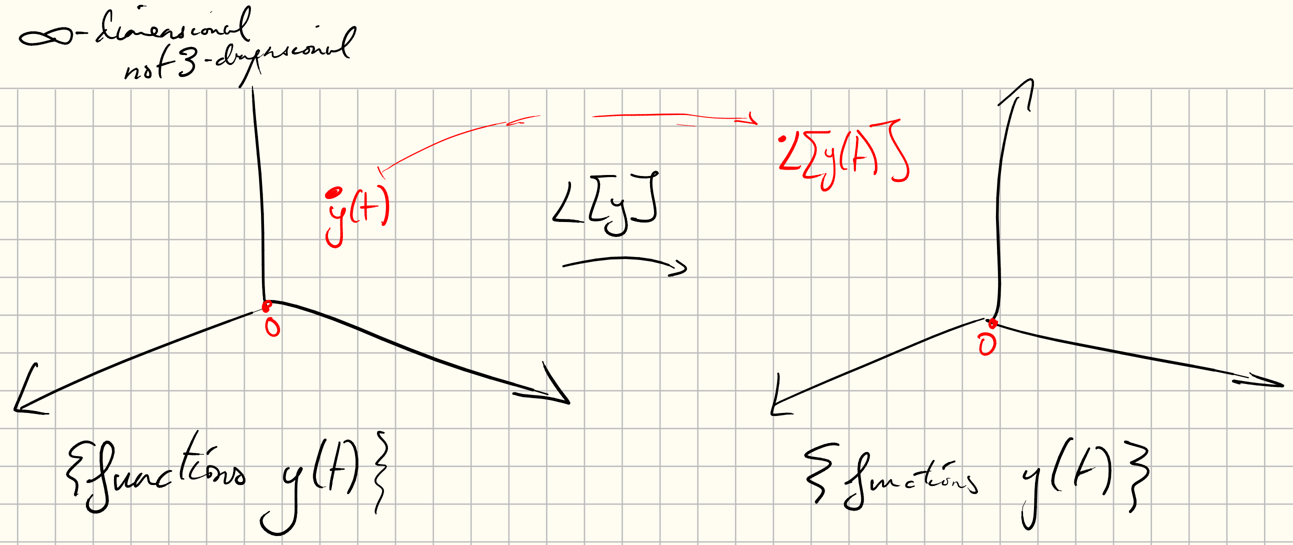
We want to replace

M with a linear differential operator $L[y]$

$\mathbb{R}^3$ with a function space

$\{\text{functions } y(t) \mid \text{satisfying some property}\}$.

There is a field of math called functional analysis that studies the different varieties of function space.

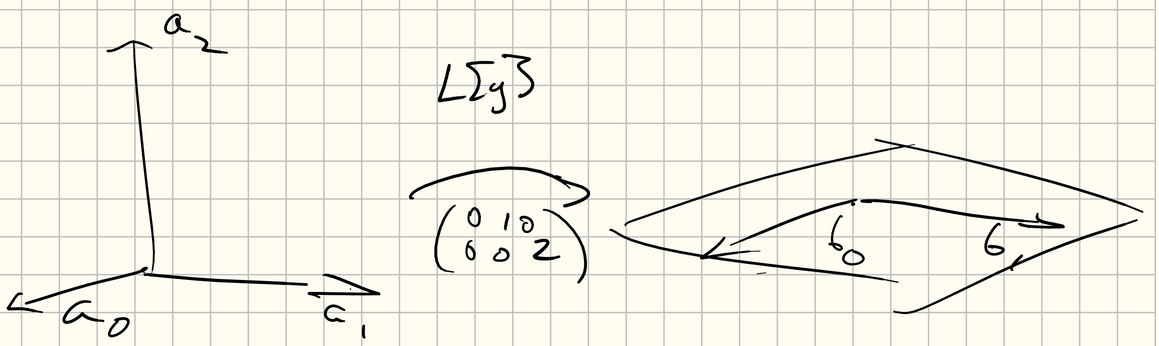


$$y(t) \xmapsto{\text{E.g.}} a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)$$

i.e. We can think of $L[y]$ as a matrix M.

This is literally true if our function space is the space of n-th order Taylor series

$$\{a_0 + a_1 t + a_2 t^2 + O(t^3)\} \xrightarrow{L[y]=\frac{d}{dt}} \{b_0 + b_1 t + O(t^2)\}$$

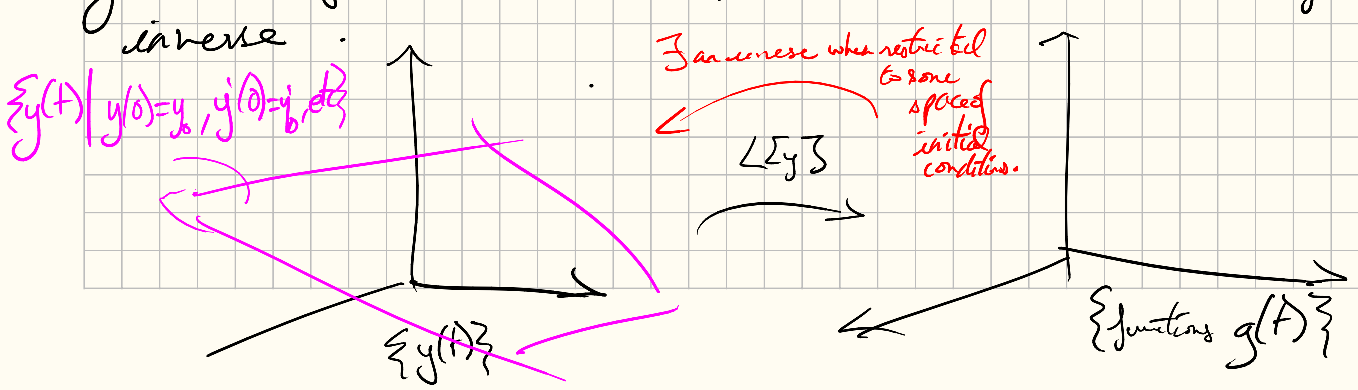


Given some $L[y]$ we would like to solve

$L[y] = g(t)$ by finding L^{-1}. E.g.:

$y(t) = L^{-1}[g]$ would be a solution.

Just like the finite dimensional case, we need some conditions to find an inverse.



We've seen this before:

If we want to solve

$$L[y] = g(t)$$ then our general solution is

$$y_p(t) + c\,y_c(t)$$ and we need to specify enough initial conditions to determine c.

From this perspective, the way to find these solutions is clear:

find "$L^{-1}[y]$" such that

$$y = L^{-1}[g(t)]$$ solves the equation.

We want a function-space version of matrix multiplication:

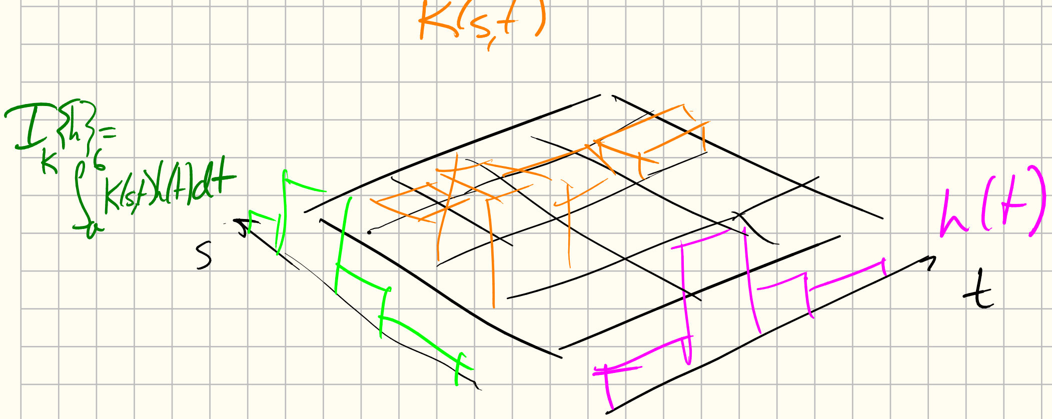
These are operators given by integral kernels.

Def: An integral operator is a linear operator of the form

$$h(t) \longmapsto \int_a^b K(s,t)\,h(t)\,dt$$

$K(s,t)$ is the integral kernel

Rmk: This is matrix multiplication on Riemann sums



$I_K(h)$ is approximated by $\sum_{i,j=1}^{4} K(a+i\Delta s,\, a+i\Delta t)\, h(a+j\Delta t)\,\Delta t \simeq I_K\{h\}$

This is matrix multiplication.

$$I_K\{h\} \simeq \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} = \begin{array}{c} j \\ i\begin{pmatrix} & & & \\ & & K(a+i\Delta s, a+j\Delta t) & \\ & & & \end{pmatrix} \end{array} \begin{pmatrix} h(a+j\Delta t) \\ \vdots \end{pmatrix}$$

$$\Big\downarrow \lim_{\Delta s, \Delta t \to 0}$$

$$= \int_a^b K(s,t)h(t)\,dt.$$

Eg: 1) $\mathcal{L}\{f(t)\} = I_{e^{-st}}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$K(s,t) = e^{-st}$

2) $K(s,t) = \delta(s-t)$ - Then we get the identity matrix

$$I_{\delta(s-t)}\{f(t)\} = \int_{-\infty}^{\infty} \delta(s-t) f(t)\,dt = (\delta * f)(s) = f(s)$$

We want the inverse of $L[y]$ subject to some initial conditions.

Make the ansatz "$L^{-1}[h(t)]$" $= \int_a^b G(t,s)\,h(s)\,ds$

Claim: If $L[G(t,s)] = \delta(t-s)$ (as a function of t) then this works, i.e.

$$y(t) = \int_a^b G(t,s)\,h(s)\,ds \quad \text{solves} \quad L[y(t)] = h(t).$$

Pf: We want to calculate $L[y(t)]$ see if it solves the equation

$$L[y(t)] = L\left[\int_a^b G(t,s)h(s)\,ds\right]$$

$$= \int_a^b L[G(t,s)]\,h(s)\,ds \qquad \text{by linearity}$$

$$= \int_0^b \delta(t-s)\, h(s)\, ds \qquad \text{by defining property of } G(s,t)$$

$$L[y(t)] = h(s) \quad \checkmark .$$

This warrants a definition

Def Given • $L[y]$ a linear differential operator $\left(\text{e.g.: } a(t)y'' + b(t)y' + c(t)y \right)$

• Some class of initial values / boundary values

A Green's function for $L[y]$ is a function

$G(s,t)$ so that

$$\boxed{L[G(s,t)] = \delta(s-t)} .$$

The argument above shows that

If $G(s,t)$ is a Green's function then

$$y(t) = \int G(s,t)\, h(s)\, ds \quad \text{solves}$$

$$L[y] = h(t) .$$

Eg: (0 initial conditions, time-invariant L)

Last time we saw that if $G(s,t) = h(t-s)$ where $h(t)$ is the impulse response then

$G(s,t)$ is a Green's function for $L[y] = a_n y^{(n)} + \dots + a_1 y' + a_0 y$.

Because $h(t)$ solves $L[y] = \delta(t)$, $y(0)=0, \dots, y^{(n-1)}(0) = 0$

$$L[h] = \delta(t) \quad \text{so}$$

$$\therefore \; y(t) = \int_0^t G(s-t)\, f(s)\, ds \quad \text{solves} \quad L[y] = f(t) .$$

This method solves

a) More general boundary conditions $\mathcal{L}[y]$

b) Linear 2nd order PDEs

Eg

heat

Wave

Potential (e.g. electrostatics)

are solved by similar methods

Eg: heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} u \qquad \text{w/ initial values}$$

$$G(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

i.e.:

$$u(x,t) = \int (G(t,x-s)\, u(s,0))\, ds$$ solves the heat equation.