ODEs 6-9

Final exam: Released Monday 6-14 after lecture
Due Wednesday 6-16 at 11:59 pm
+ 24 grace period

Goal:
- Boundary value problems
- Diagonalize 2nd order differential operators on function spaces (Sturm-Liouville theory).

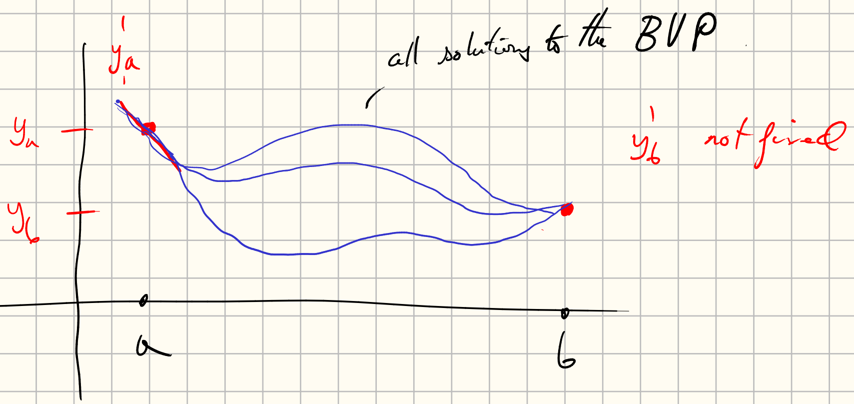
A boundary value problem for an ODE $L[y] = g(x)$ where $L[y]$ is a linear differential operator e.g. $L[y] = a(t)y'' + b(t)y' + c(t)y$

$$L[y] = 0$$

on an interval $[a,b]$ is a specification of

$$y(a) = y_a \quad y'(a) = y'_a$$
$$y(b) = y_b \quad y'(b) = y'_b$$

or some subset of these.



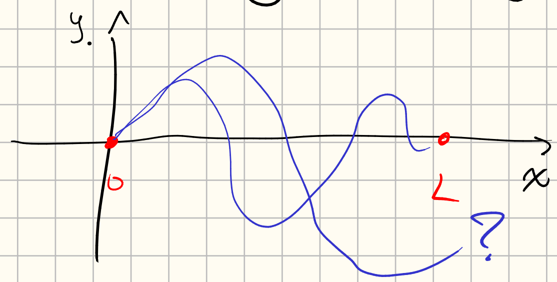
Ideal situation: If we fix
- n equations for boundary values
- $L[y]$ is order n

we hope to find a unique solution to the BVP.

It doesn't work out like this, there could be
0, 1, or many solutions to a specific BVP.

Eg: $y'' + \omega^2 y = 0$ with BVP on $[0, L]$

$y(0) = 0 \quad y(L) = 0$



The general solution to this ODE is
$$y(x) = c_1 \sin(\omega x) + c_2 \cos(\omega x).$$
The boundary conditions give equations for c_1 & c_2.

$y(0) = 0$ so
$$c_1 \sin(0) + c_2 \cos(0) = 0$$
$y(L) = 0$ so
$$c_1 \sin(\omega L) + c_2 \cos(\omega L) = 0$$

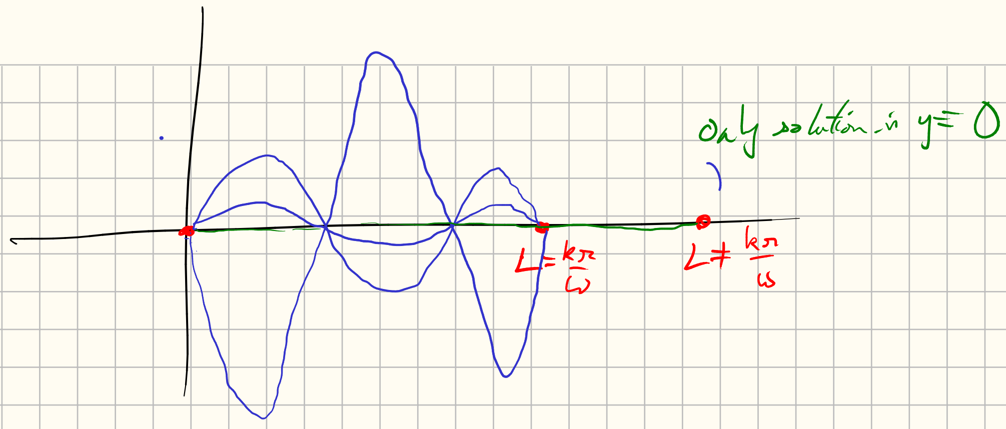
$$c_1 0 + c_2 1 = 0 \Rightarrow c_2 = 0$$

$$c_1 \sin(\omega L) = 0 \Rightarrow 2 \text{ case}$$

Case 1: $c_1 = 0$
The solution is $y(t) \equiv 0$.

Case 2: $\sin(\omega L) = 0$
can only be true if $\omega L = k\pi$ $k \in \mathbb{Z}$, or
$$\omega = \frac{k\pi}{L}.$$
But if ω is an integer multiple of $\frac{\pi}{L}$ then
$$y(t) = c_1 \sin(\omega t)$$ is a solution for any c_1

i.e. many solutions (doesn't happen for IVPs)



Eg: Solve the BVP on $[0,1]$

$$y'' - y = 0 \qquad y(0) = 0 \quad y(1) = 1$$

General solution is

$$c_1 e^{t} + c_2 e^{-t}$$

$y(0) = 0$:

$$c_1 e^0 + c_2 e^0 = 0 \quad \text{or} \quad c_1 + c_2 = 0$$

$y(1) = 1$

$$c_1 e^1 + c_2 e^{-1} = 1$$

$$c_1 \left(e - \frac{1}{e}\right) = 1 \qquad \Big\} \; c_2 = -c_1$$

$$c_1 = \frac{1}{e - \frac{1}{e}}$$ and our unique solution is

$$y(t) = \frac{1}{e - \frac{1}{e}} e^{t} - \frac{1}{e - \frac{1}{e}} e^{-t}.$$

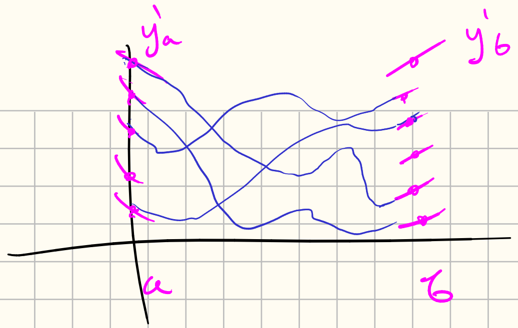
There are names for different types of boundary conditions.

Def: Dirichlet Boundary conditions we specify

$y(a) = y_a \qquad y(b) = y_b$ at the endpoints of $[a,b]$

Def: Neumann boundary conditions we specify

$y'(a) = y_a' \qquad y'(b) = y_b'$ at endpoints of $[a,b]$



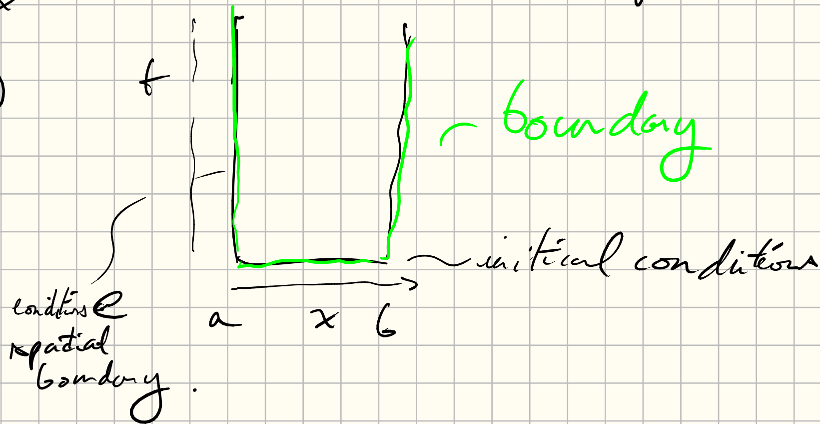
Eg (Heat equation)

Rmk: These have definitions for PDEs as well as ODEs

$$\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u$$ is the 1d heat equation

on $t \in [0,\infty)$

$x \in [a,b]$

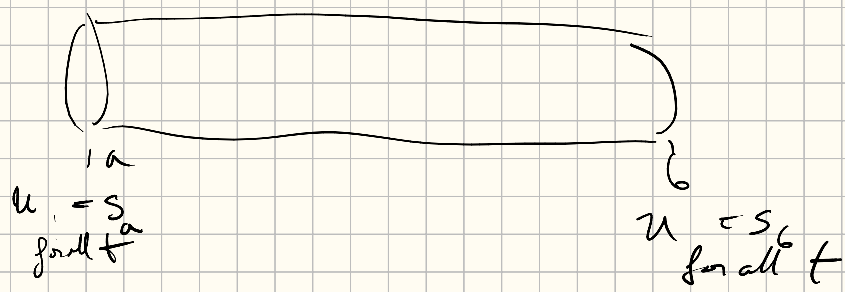


Dirichlet conditions at spatial boundary $x=a$, $x=b$

$u(a,t) = s_a$

$u(b,t) = s_b$

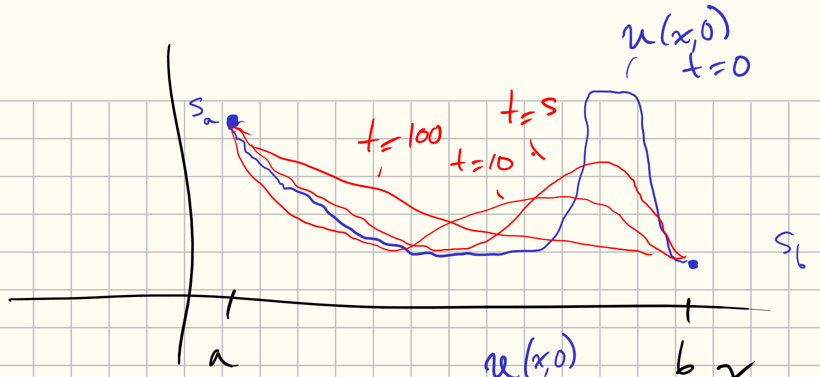
Metal rod with temperature fixed at both ends



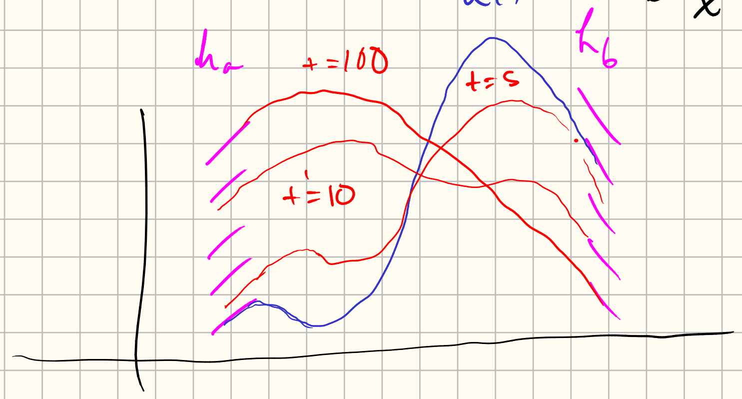
Neumann boundary conditions fix the amount that we heat each end.

$\frac{\partial}{\partial x} u(x,t)\Big|_{x=a} = h_a$ $\qquad$ $\frac{\partial}{\partial x} u(x,t)\Big|_{x=b} = h_b$

Eg Dirichlet boundary conditions



Neumann b/c



Recall: The analogy between

vector spaces	:	function spaces
Matrices	:	linear operators
eigenvalues	:	?
eigenvectors	:	?
basis	:	?

These only exist in special cases, and we have to use more theory.

We want to study a very special type of linear differential operator where we can fill in the ? marks.

§ Sturm-Liouville boundary value problems

When the function space is functions on $[a,b]$ w/ some properties

– Linear operator is $a(t)y'' + b(t)y' + c(t)y = L[y]$

we can actually do this.

Idea: If we squint

$$a(t)y'' + b(t)y' + \lambda c(t)y = 0$$ is the eigenvalue equation.

Write it as

$$-a(t)y''-b(t)y' = \lambda c(t)y$$

OR if $L[y]=-a(t)y''-b(t)y'$

$$\boxed{L[y]=\lambda c(t)y}$$

i.e. $y(t)$ is an "eigenvector" of $L[y]$ with "eigenvalue" $\lambda c(t)$

Rewrite $L[y]=-(p(t)y')'+q(t)y'$ (we can always do this)

Def: A Sturm-Liouville BVP is one of the type

$$L[y]=\lambda c(t)y \quad \text{where } L[y]=-(p(t)y')'+q(t)y'$$

with boundary conditions on $[a,b]$

$$\alpha_0 y(a)+\alpha_1 y'(a)=0 \qquad \beta_0 y(b)+\beta_1 y'(b)=0$$

Assume: $p(t)>0$ on $[a,b]$ and $c(t)>0$ on $[a,b]$
then it is a Regular SL-problem.

Eg: $y''=-\omega^2 y$ on $[0,L]$ $y(0)=0$ $y(L)=0$ is a SL problem.

Rmk: This example is the prototype: general theory generalizes the following facts about $\sin(\omega t)$ functions on $[0,L]$

1) There is a sequence $\omega=\frac{k\pi}{L}$ $k=0,1,2,\dots$
where the BVP

$$y''-\omega^2 y; \quad y(0)=0 \quad y(L)=0 \text{ has a non-zero solution}$$

these solutions ω^2 are the eigenvalues and
$\sin(\omega t)$ are eigenfunctions

2) We have Fourier series: i.e. we can write a function f on $[0,L]$ with $f(0)=0$, $f(L)=0$ as a sum of $\sin(\frac{k\pi}{L}t)$:

$$f(t) = \sum_{k=0}^{\infty} a_n \sin\left(\frac{k\pi}{L} t\right)$$

Slogan: "We can diagonalize $L[y]$ on the space of weighted square integrable functions".

Def: If $[a,b]$ is an interval, $c(t)$ is a function of $[a,b]$ then $L^2_{c(t)}([a,b])_{bc}$ is the the function space

$$L^2_{c(t)}([a,b])_{bc} = \left\{ \begin{array}{c} \text{functions on} \\ [a,b] \end{array} \middle| \begin{array}{l} \bullet \int_a^b c(t) f^2(t)\, dt < \infty \\ \bullet\ f(t) \text{ satisfies the boundary conditions} \end{array} \right\}$$

Thm: (Solution to Sturm-Liouville problems)

Given
$$\boxed{L[y] = \lambda c(t) y \quad \text{on } [a,b] \quad \begin{array}{l} \alpha_0 y(a) + \alpha_1 y'(a) = 0 \\ \beta_0 y(b) + \beta_1 y'(b) = 0 \end{array}}$$

there is an infinite sequence of eigenvalues and eigenfunctions

$$\boxed{\begin{array}{l} \lambda_1 < \lambda_2 < \lambda_3 \leq \cdots \\ \phi_1, \phi_2, \phi_3 \cdots \end{array}}$$

so that
- $L[\phi_i] = \lambda_i c(t) \phi_i$.
- $\{\phi_1, \phi_2, \ldots\}$ is a (Hilbert space) basis for the vector space $L^2_{c(t)}([a,b])_{bc}$

Eg: ($\sin(\omega t)$ Fourier series)

- $L[y] := -y''$ on the basis $\{\underbrace{\sin\left(\frac{k\pi}{L}t\right)}_{\phi_k}\}_{k=0,1,\dots}$

$L[y]$ is represented by the matrix:

$$
\begin{array}{c|c}
 & \begin{matrix} \phi_0 & \phi_1 & \phi_2 & \phi_3 & \cdots \end{matrix} \\
\begin{matrix} \phi_0 \\ \phi_1 \\ \vdots \end{matrix} &
\begin{pmatrix}
0 & & & & \\
 & \left(\frac{\pi}{L}\right)^2 & & 0 & \\
 & & \left(\frac{2\pi}{L}\right)^2 & & \\
 & 0 & & \left(\frac{3\pi}{L}\right)^2 & \\
 & & & & \left(\frac{4\pi}{L}\right)^2 & \ddots
\end{pmatrix}
\end{array}
$$

Eg: Fourier-Bessel series of order 0. (This is actually singular, but the result still holds).

(Mode expansion of waves in cylindrical coordinates)

Bessel eq. for $x \mapsto \sqrt{\lambda}x$

$$-(xy')' + \frac{\nu^2}{x}y = \lambda x y \qquad \text{OR}$$

$$-(xy')' = \lambda x y \qquad \text{ON } [0,1] \quad y(0) = y(1) = 0$$

Then eigenvalues

$$\lambda_0, \lambda_1, \lambda_2, \dots$$

& eigenfunctions

$$J_0(\sqrt{\lambda_0}x), J_0(\sqrt{\lambda_1}x), J_0(\sqrt{\lambda_2}x) \dots$$

So that for $f(x)$ appropriate

$$f(x) = \sum_{k=0}^{\infty} a_k J_0(\sqrt{\lambda_k}x)$$

In this basis $L[y]$ has matrix

$$
\begin{array}{c|c}
 & \phi_0 \quad \cdots \quad \phi_k \quad \cdots \\
\hline
\phi_0 & \begin{pmatrix} x\sqrt{\lambda_0} & & & & & 0 \\ & x\sqrt{\lambda_1} & & & & \\ & & \ddots & & & \\ & & & x\sqrt{\lambda_k} & & \\ & 0 & & & \ddots & \\ & & & & & \ddots \end{pmatrix}
\end{array}
= L[y].
$$

Rmk: There is a formula for a_k:

$$\boxed{a_k = \int_a^b f(t)c(t)\,\phi_k(t)\,dt}$$

Eg: Any $f(t)$ w/ $f(0)=f(L)=0$ satisfies

$$f(t) = \sum_{n=0}^{\infty} a_n \sin\left(\frac{k\pi}{L}t\right) \quad \text{where}$$

$c(t)=1$

$$a_n = \int_0^L f(t)\sin\left(\frac{k\pi}{L}t\right)dt.$$