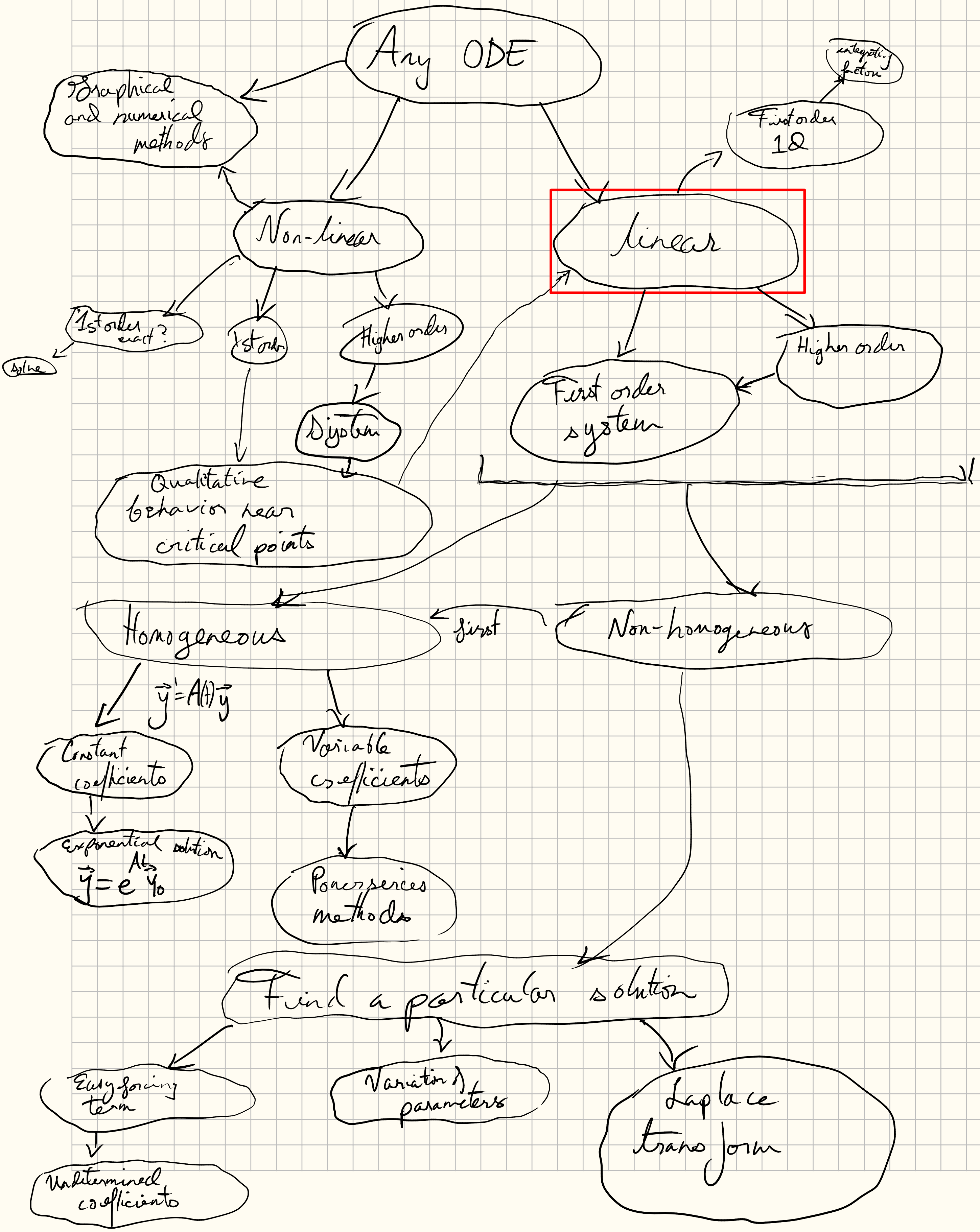
ODEs 6-10

Final review

How do we solve an ODE?



Examples

Eg: Find the general solution to

$$ty' - y = t^2 e^{-t}$$

It's 1st order linear DE : Use an integrating factor

$$y' - \frac{1}{t}y = te^{-t} \qquad y' + p(t)y = q(t)$$

Integrating factor: $\mu(t) = e^{\int p(t)dt} = e^{\int -\frac{1}{t}dt} = e^{-\log(t)} = \frac{1}{t}$

$$\mu(t)\left[y' - \frac{1}{t}y\right] = \mu(t)(te^{-t})$$

$$\frac{1}{t}y' - \frac{1}{t^2}y = e^{-t}$$

LHS is a total derivative

$$\left(\frac{1}{t}y\right)' = e^{-t}$$

$$\frac{1}{t}y = \int e^{-t}dt + c$$

$$y = t\left(\int e^{-t}dt + c\right)$$

$$\boxed{y(t) = ct - te^{-t}}$$

Eg: Find the general solution to

$$y^{(3)} - y'' - y' + y = 0$$

Two options: 1) Rewrite as 3d system and solve that way

2) Equivalent: use techniques based on the ansatz $y(t) = e^{rt}$.

Characteristic polynomial:

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

$\lambda = 1$ is a solution

$$\begin{array}{r|l} & \lambda^2 - 1 \\ \lambda - 1 & \lambda^3 - \lambda^2 - \lambda + 1 \\ & \underline{\lambda^3 - \lambda^2} \\ & 0 - \lambda + 1 \\ & \underline{-\lambda + 1} \\ & 0 \end{array}$$

$$(\lambda^2 - 1) = (\lambda - 1)(\lambda + 1)$$

Roots: $\underline{1, 1, -1}$.

Solution: $\boxed{c_1 e^{-t} + c_2 e^{t} + c_3 t e^{t}}$.

Which is the first row of

$$c_1 e^{-t} \vec{\xi}_{-1} + c_2 e^{t} \vec{\xi}_1 + c_3 \left(t e^{t} \vec{\xi}_1 + e^{t} \vec{\eta}_1 \right).$$

$\vec{\xi}_{-1}, \vec{\xi}_1, \vec{\eta}_1$ are generalized eigenvectors of

$$\begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \end{pmatrix}$$

$$\ddot{y}' = -y + \dot{y} + \ddot{y}$$

Eg: Find the general solution to

$$\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$$

and classify the critical point at $\vec{x} = 0$.

Characteristic polynomial of $\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

$$\det \begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix} = (2-\lambda)(-2-\lambda) + 5 = \underline{\lambda^2 + 1}$$

with roots $\pm i$

$\underline{\lambda = i:}$ $\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \xrightarrow{r_2 \mapsto r_2 - (2-i) r_1} \begin{pmatrix} 2-i & -5 \\ 0 & (-2-i)+5(2-i) \end{pmatrix} = \begin{pmatrix} 2-i & -5 \\ 0 & 8-6i \end{pmatrix}$$

$a = 5 \quad b = 2-i$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} 5 \\ 2-i \end{pmatrix} = \begin{pmatrix} 0 \\ 5+(2-i)(-2-i) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{\xi}_i = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

$\underline{\text{General solution:}}$ $c_1 \operatorname{Re}\left(e^{it} \vec{\xi}_i\right) + c_2 \operatorname{Im}\left(e^{it} \vec{\xi}_i\right) \quad c_1, c_2 \in \mathbb{R}$

$$e^{it} \vec{\xi}_i = \left(\cos(t) + i \sin(t)\right) \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

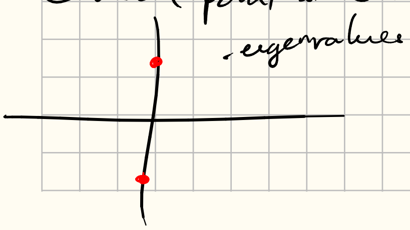
$$= \begin{pmatrix} 5\cos(t) + i\, 5 \sin(t) \\ 2\cos(t) + \sin(t) + i\left(-\cos(t) + 2\sin(t)\right) \end{pmatrix}$$

$$= \begin{pmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} + i \begin{pmatrix} 5\sin(t) \\ -\cos(t) + 2\sin(t) \end{pmatrix}$$

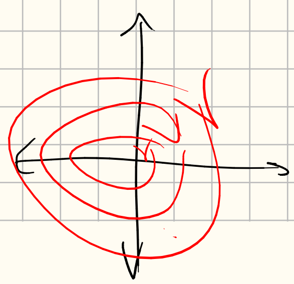
$\underline{\text{Solution:}}$

$$\boxed{c_1 \begin{pmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} + c_2 \begin{pmatrix} 5\sin(t) \\ -\cos(t) + 2\sin(t) \end{pmatrix}}$$

Critical point at 0:



$\Rightarrow$ $\vec{x} = 0$ is a cycle



Eg: Describe the qualitative behavior of the competing species model

$$x' = x(1.5 - x - 0.5y) \qquad x, y \geq 0$$
$$y' = y(2 - y - 0.75x)$$

Find critical points:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} 0 &= x(1.5 - x - 0.5y) \\ 0 &= y(2 - y - 0.75x) \end{aligned}$$

1) $x=0$ $y=0$ 2) $x=0,\ y=2$ 3) $x=1.5,\ y=0$

$0 = y(2-y)$

4) $x \neq 0, y \neq 0$, $(1.5 - x - 0.5y) = 0$ and

$(2 - y - 0.75x) = 0$

or

$$\begin{pmatrix} 1 & 0.5 \\ 0.75 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\frac{5}{8}} \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{3}{4} & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ 2 \end{pmatrix}$$

$$\frac{8}{5} \begin{pmatrix} \frac{1}{2} \\ -\frac{9}{8} + 9 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{63}{5} \end{pmatrix}$$

To find the qualitative behavior, we calculate $J_F(\text{critical point})$ where J_F is the Jacobian

$$F(x,y) = \begin{pmatrix} x(1.5 - x - 0.5y) \\ y(2 - y - 0.75x) \end{pmatrix} - \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

$$J_F(x,y) = \begin{pmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_x}{\partial y} \\ \frac{\partial F_y}{\partial x} & \frac{\partial F_y}{\partial y} \end{pmatrix} = \begin{pmatrix} 1.5 - 2x - 0.5y & -0.5x \\ -0.75y & 2 - 2y \end{pmatrix}$$

1) $x=y=0$

$J_F = \begin{pmatrix} 1.5 & 0 \\ 0 & 2 \end{pmatrix}$

2) $x=0$, $y=2$

$J_F = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & -2 \end{pmatrix}$

3) $x=1.5$ $y=0$

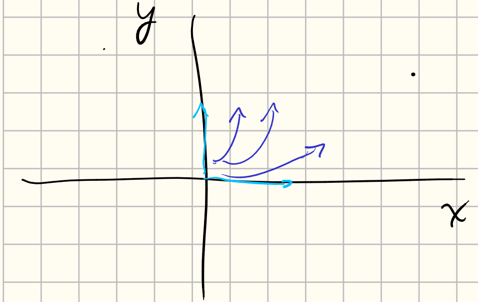
$J_F = \begin{pmatrix} -1.5 & -\frac{3}{4} \\ 0 & 2 \end{pmatrix}$

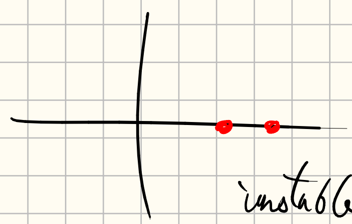
4) $x=\frac{4}{5}$ $y=\frac{63}{5}$

$J_F = J_F\left(4/5,\ 63/5\right)$

Classify critical points:

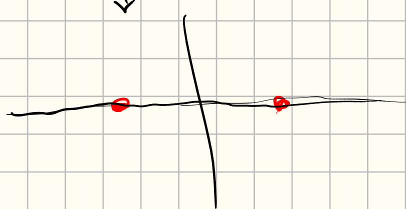
1) $J_F = \begin{pmatrix} 1.5 & 0 \\ 0 & 2 \end{pmatrix}$ eigenvalues 1.5 and 2





unstable node

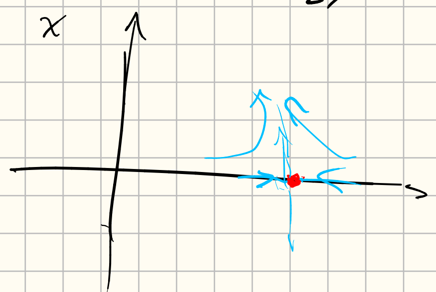
saddle



2) $J_F = \begin{pmatrix} \frac{1}{2} & 0 \\ -3/2 & -2 \end{pmatrix}$

$\det\begin{pmatrix} \frac{1}{2}-\lambda & 0 \\ 3/2 & -2-\lambda \end{pmatrix} = \left(\frac{1}{2}-\lambda\right)(-2-\lambda)$

with roots $\lambda = -2, \frac{1}{2}$

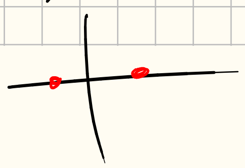


3) and 4)

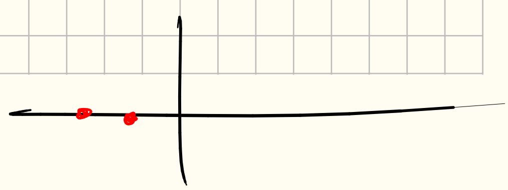
3) $x=1.5$, $y=0$

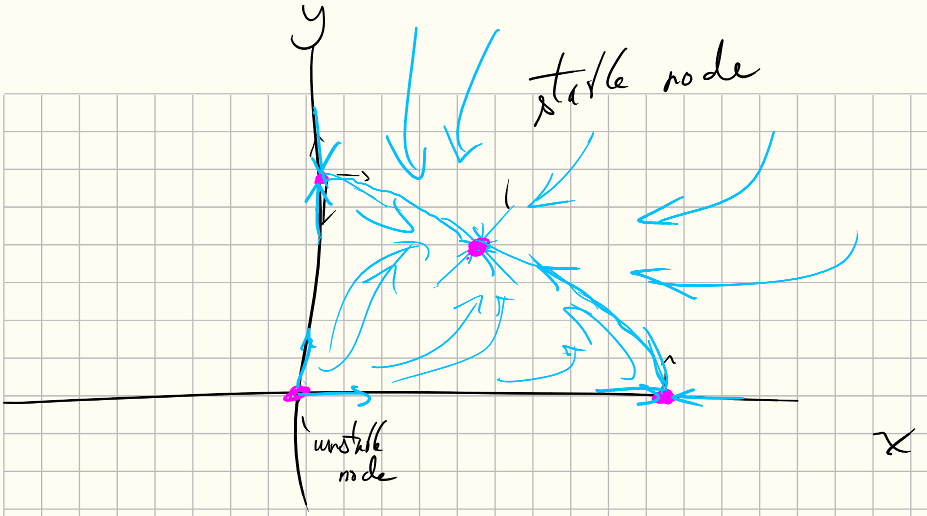
symmetric w/ x by

saddle



4) $x\neq 0$, $y\neq 0$ stable node





Eg: Find a series solution to the Legendre equation

$$(1-x^2)y'' - 2x y' + \alpha(\alpha+1)y = 0 \quad \text{around } x=1.$$

Notice: $x=1$ is a singular point: plugging in $x=1$: We can't solve for y''.

Use the Frobenius method: Substitute $t = x-1$. $\frac{d}{dt} = \frac{d}{dx}$

Now we seek a series around $t=0$

$$t^r\sum_{n=0}^{\infty} a_n t^n \quad \text{instead of} \quad (x-1)^r\sum_{n=0}^{\infty} a_n (x-1)^n.$$

$$(1-(t+1)^2)y'' - 2(t+1)y' + \alpha(\alpha+1)y = 0$$

$$(t(t+2))y'' - 2(t+1)y' + \alpha(\alpha+1)y = 0$$

Regular singular point:

$$\lim_{t\to 0} \frac{t\,2(t+1)}{t(t+2)} = \frac{2(0+1)}{(0+2)} = 1 \neq \infty = p(0) \quad \checkmark$$

$$\lim_{t\to 0} \frac{t^2(\alpha(\alpha+1))}{t(t+2)} = 0 \neq \infty = q(0) \quad \checkmark$$

Indicial equation:

$$r(r-1) + p(0)r + q(0) = r^2 - r + r = r^2, \quad \text{roots } r = 0, 0.$$

We look for series solutions of the form

$$y_1 = t^0 \sum_{n=0}^{\infty} a_n t^n \quad \text{and} \quad C\log(t)\, y_1 + \sum_{n=0}^{\infty} b_n t^n$$