Homework #1

MATH 2030 Summer A 2021

Due: May 6, 2021

Problem 1

For each of the following ordinary differential equations

(i)
$$\frac{dy}{dt} = -\frac{t}{y}$$

(ii)
$$\frac{dy}{dt} = t - 2y$$

do the following: a) Plot a sketch of the slope field for the differential equation. b) On the same plot, draw three different integral curves for different initial conditions.

Problem 2

For each of the following differential equations, answer the following questions: a) What is the order of the equation? b) Is the equation linear or nonlinear? c) Is the equation autonomous?

(i) $\frac{dy}{dt} + \sin(t)y^2 = 0$

(ii) $\frac{d^3y}{dt^3} + 10\frac{d^2y}{dt^2} - 3y = 0$

(iii) $\frac{d^2y}{dt^2} + y^3\frac{dy}{dt} + \cos(y) = 0.$

Problem 3

(i) The equation

$$\frac{dP}{dt} = rP(1 - \frac{P}{K})$$

is the logistic equation, which is a simple model for population growth, describing the dynamics over time t of a population P with growth rate r and carrying capacity K.

Verify that

$$P(t) = \frac{KP_0e^{rt}}{K + P_0(e^{rt} - 1)}$$

is the general solution to this equation.

(ii) a) Verify that the differential equation

$$y' = \cos(t)y$$

has general solution $y(t) = c\exp(\sin(t))$ for a constant c. b) More generally, verify that the differential equation $y' = a(t)y$ has general solution $y(t) = c\exp(\int a(t)dt)$ where $\int a(t)dt$ refers to any antiderivative of $a(t)$.

Problem 4

Find the general solution of the following differential equations (you may leave your answer as an implicit solution)

(i) $y'(t) = 4t^2y^3$

(ii) $y'(x) = \frac{e^{-x}}{y+e^y}$

(iii) $y'(t) = te^{4t}$.

Problem 5

Solve the following IVPs. You may leave your solution as an implicit equation.

(i) $y' = -t/y, \quad y(0) = 5$

(ii) $y' = t^2y, \quad y(0) = 1$. (Hint: you can use problem 3)(ii) part b).

Problem 6

Newtonian mechanics is widely understood to be a *deterministic* description of physics. Mathematically this is related to the uniqueness part of the existence/uniqueness theorem for ODEs, where the ODE form of Newton's second law for an object of mass m moving in direction y subject to a force F is

$$\frac{d^2y}{dt^2} = \frac{F}{m}.$$

There is a description of an object sitting on top of a pointy (discontinuous) hill called *Norton's dome* which subjects an object to a force tangent to the dome equal to $F = m\sqrt{|y|}$ where y is the distance from the tip of the dome. a) Find constants c_0 and α such that for any time t_M the differential equation

$$\frac{d^2y}{dt^2} = \sqrt{|y|}$$

has continuous, twice differentiable solution

$$y(t) = \begin{cases} 0, & 0 \le t \le t_M \\ c_0(t - t_M)^{\alpha}, & t \ge t_M. \end{cases}$$

b) Why does this not violate the existence and uniqueness theorem, despite the fact that the IVP $y(0) = 0, y'(0) = 0$ has infinitely many solutions, one for each $t_M \ge 0$?

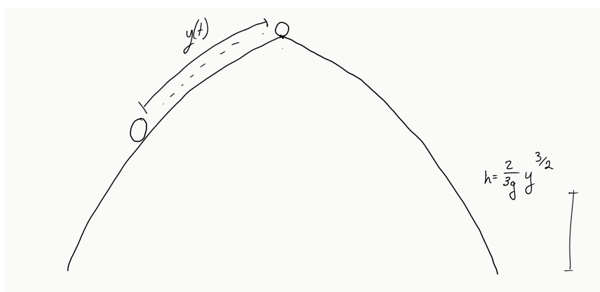

Figure 1: Norton's dome, with distance along the dome $y(t)$, such that it's height is $2/3gy^{3/2}$ where g is the acceleration due to gravity.