Homework #3

MATH 2030 Summer A 2021

Due: May 14, 2021

Problem 1

For each of the following equations: a) Write the general complex solution, with complex coefficients. Clearly state that your coefficients are complex numbers, e.g. write $d \in \mathbb{C}$ for the coefficient d. b) Write the general real solution, with real coefficients. Clearly state that your coefficients are real numbers, e.g. write $d \in \mathbb{R}$ for the coefficient d. c) Calculate the Wronskian of a fundamental set of real-valued soltuions. Conclude that your solution is the general solution to the equation.

(i) $y'' + 5y' + 6y = 0$.

(ii) $y'' + y' + 2y = 0$.

(iii) $y''' + 5y'' + 17y' + 13y = 0$.

Problem 2

The **Euler equation** is the 2nd order ODE

$$t^2 \frac{d^2y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0, \ t > 0$$

for real constants $\alpha, \beta \in \mathbb{R}$. This does not have constant coefficients but can be transformed into a constant coefficient equation via the change of variables $x = \log(t)$. The method of **change of variables** frequently allows you to convert a more challenging problem into a simpler one.

(i) Let $x = \log(t)$. Calculate d^2y/dt^2 in terms of x and d^2y/dx^2, and calculate dy/dt in terms of x and dy/dx.

(ii) Show that the Euler equation can be rewritten in the form

$$\frac{d^2y}{dx^2} + \gamma \frac{dy}{dx} + \delta y = 0$$

which has constant coefficients, where $\gamma, \delta \in \mathbb{R}$ are real constants. Find expressions for γ and δ in terms of α and β.

(iii) Use the previous parts of the problem to solve the following IVP for a (real-valued) funtion $y(t), t > 0$:

$$t^2 y''(t) + 7ty'(t) + 10y(t) = 0, \quad y(1) = 0, y'(1) = 1.$$

Make sure your answer is written in terms of the variable t.

Problem 3

Using the method of undetermined coefficients, find the general solution to the following ODE. (You may use your answer to Problem 1).

$$y'' + y' + 2y = cos(2t) + t^2.$$

Problem 4

Consider the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, y'(0) = y_0'.$$

(i) Show that

$$\lim_{t\to\infty} y(t) = \begin{cases} +\infty & \text{if } y_0' \geq 2 \\ -\infty & \text{if } y_0' < 2 \end{cases}.$$

(ii) Now we will consider the problem where we slightly deform the y' coefficient of the equation to obtain the IVP

$$y'' + (-4 + \epsilon)y' + 4y = 0, \quad y(0) = 1, y'(0) = y_0'.$$

Show that if $\epsilon > 0$ is arbitrarily small then

$$\lim_{t\to\infty} y(t)$$

does not exist for any value of y_0'.

It may be instructive to also consider the cases $\epsilon < 0$ and other values of y_0.

Problem 5

For the following matrices A calculate a) The eigenvalues and eigenvectors of A. b) The inverse A^{-1} of the matrix.

(i)

$$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

(ii)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}.$$