Homework #4

MATH 2030 Summer A 2021

Due: May 17, 2021

Problem 1

i) Using Euler's formula, prove the following identities

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

ii) Define the *hyperbolic trigonometric functions* by the formulas

$$\cosh(x) = \frac{e^{-x} + e^{x}}{2}$$
$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}.$$

Show that these functions satisfying the following hyperbolic version of the Pythagorean theorem:

$$\cosh^2(x) - \sinh^2(x) = 1$$

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Problem 2

i) Verify that the vector-valued function $\mathbf{x}(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$ is a solution to the 1st order system of equations $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

ii) Verify that the vector-valued function $\mathbf{x}(t) = \begin{pmatrix} \sinh(t) \\ \cosh(t) \end{pmatrix}$ is a solution to the system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Problem 3

For each of the following matrices A, find the general solution to the first order system $\mathbf{x}' = A\mathbf{x}$.

i) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

ii) $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$.

Problem 4

Let $A(t)$ be an $n \times n$ matrix whose coordinates are functions of t. Prove the *principle of superposition for linear systems*, which is the following statement: If $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$ are two solutions to the system of equations

$$\mathbf{x}' = A(t)\mathbf{x}$$

then so is $c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$ for any constants c_1 and c_2.

Problem 5

Solve the following initial value problem, and descibre the behavior of this solution as $t \to \infty$.

$$\mathbf{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}.$$