# Homework #5

MATH 2030 Summer A 2021

Due: May 24, 2021

## Problem 1

Find the general real-valued solution to system $\mathbf{x}' = A\mathbf{x}$ when $A$ is the matrix.

i) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

ii) $A = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & 1 \\ 0 & 0 & 0 & 0 & 0 & -8 \end{pmatrix}.$

## Problem 2

Configurations of eigenvalues $\{\lambda_1, \ldots, \lambda_3\}$ and $\{\lambda'_1, \lambda'_2, \lambda'_3\}$ produce qualitatively equivalent systems if we can move the corresponding complex numbers $\{\lambda_1, \lambda_2, \lambda_3\}$ around until they coincide with $\{\lambda'_1, \lambda'_2, \lambda'_3\}$ a) without any pair of points colliding or separating and b) without the real parts $\mathrm{Re}(\lambda_i)$ colliding or separating, or colliding or separating with 0.

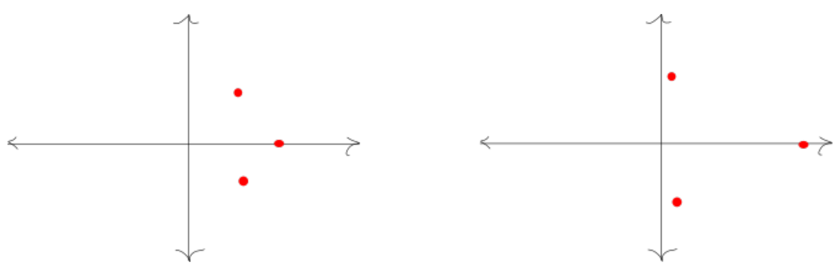
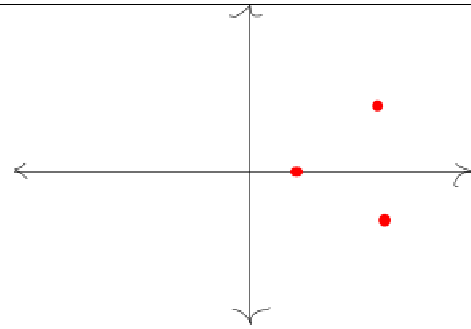
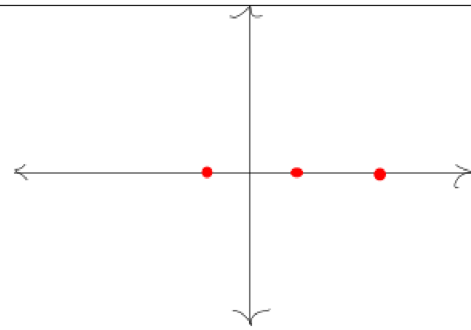


Figure 1: Equivalent configurations of eigenvalues

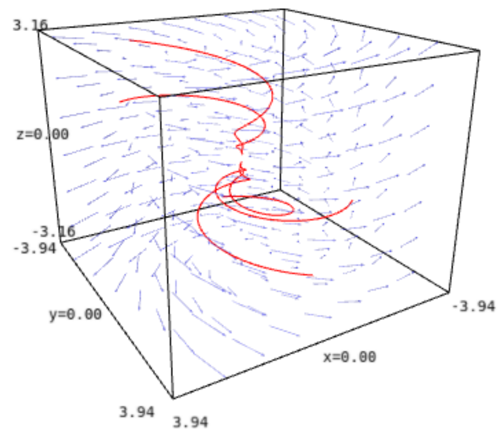
If we restrict attention to **non-degenerate** systems (i.e. those where there is no eigenvalue $\lambda = 0$) and to the cases where there are no repeated eigenvales $\lambda_i = \lambda_j$ then there are 14 different inequivalent configurations of eigenvalues. I have drawn two below. Find the other 12.

| $\{\lambda_1, \lambda_2, \lambda_3\}$ | Eigenvalues plot |
|---|---|
| $r_1, r_2 + i\omega, r_2 - i\omega$<br>$0 < r_1 < r_2$ |  |
| $r_1, r_2, r3$<br>$r_1 < 0 < r_2 < r_3$ |  |
| $\ldots$ | |
| | |

# 1 Problem 3

There are 7 different configurations which are not related by reflecting across the imaginary axis.

For each of these 7 configurations, come up with $3 \times 3$ matrix wose eigenvalues have this configuration and plot a sketch of some of the solution curves (and vector fields if you want). The `3d Homogeneous Constant Coefficients Systems` notebook will be helpful for these plots. I have done the first one:

| $\{\lambda_1, \lambda_2, \lambda_3\}$ | Matrix | Solutions |
|---|---|---|
| $r_1, r_2 + i\omega, r_2 - i\omega$<br>$0 < r_1 < r_2$ | $\begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ |  |
| ... | | |

## Problem 4

The same method readily classifies the qualitative behavior of linear systems in any dimensions, although the number of configurations of eigenvalues increases rapidly. The only fundamentally new behavior that arises for $n > 3$ is *repeated complex eigenvalues*. Find the general solution to the following linear system

$$\mathbf{x}' = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathbf{x}.$$

Hint: You should not have to do any Gaussian elimination to solve this problem. First notice that this matrix is the block matrix

$$A = \left( \begin{array}{c|c} C & I \\ \hline 0 & C \end{array} \right)$$

where $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which we have studied extensively, and know its eigenvectors and eigenvales.

## Problem 5

Using the method of variation of parameters, find the general solution of the system of equations

$$\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \qquad t > 0.$$