Homework #6

MATH 2030 Summer A 2021

Due: May 31, 2021

Problem 1

Find all critical points of the nonlinear system

$$\begin{cases} \frac{dx_1}{dt} &= x_1 x_2 + 12 \\ \frac{dx_2}{dt} &= x_1^2 + x_2^2 - 25 \end{cases}.$$

a) Determine whether each is a node, spiral, saddle, cycle, source, sink, or degenerate node and b) classify each as stable or unstable.

Problem 2

Consider the system, (the 2d **Lotka-Volterra equations**)

$$\begin{cases} \frac{dx}{dt} &= x(\alpha - \beta y) \\ \frac{dy}{dt} &= y(-\gamma + \delta x) \end{cases}, \qquad x, y \geq 0$$

which describes predator prey dynamics. Here $\alpha, \beta, \gamma, \delta > 0$ are real parameters, x is the prey population and y is the predator population (hence $x \geq 0, y \geq 0$).

i) For the specific choices $\alpha = 3/2, \beta = 1/2, \delta = 1, \gamma = 1/2$ find all critical points of the system in the quadrant $x \geq 0, y \geq 0$ and classify them as in Problem 1. If the point lies on the boundary of the quadrant, classify it as if the equation made sense for all x and y.

ii) Use what you calculated in part i) to plot the phase plane of the system with trajectories.

Problem 3

The equation

$$y'' - 2xy' + \lambda y = 0$$

where λ is a constant, and x is defined for $-\infty < x < \infty$ is known as the **Hermite equation**.

i) Write a recurrence relation for the coefficients of a series solution to this equation.

ii) Find two particular solutions to the equation and calculate the first four terms. Write your answers as

$$y_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + O(x^4)$$

and similarly for $y_2(x)$. Evaluate the Wronskian at a chosen x_0 to determine this is a fundamental set of solutions.

iii) Observe that if λ is a nonnegative even integer, then one or the other of the series solutions terminates and becomes a polynomial. Find the polynomial solutions for $\lambda = 0, 2, 4, 6, 8$ and 10. Each polynomial is determined only up to a multiplicative constant.

iv) The Hermite polynomial $H_n(x)$ is defined as the polynomial solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x_n is 2^n . Find $H_0(x), H_1(x), \ldots, H_5(x)$.

Problem 4

Consider the **Bessel equation** of order zero

$$x^2y'' + xy' + x^2y = 0.$$

i) Find a regular singular point of this equation for a finite value of x and find the roots of the indicial equation at that point.

ii) Show that one series solution for this equation is

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

and that this series converges for all x. This function is known as the Bessel function of the first kind of order 0.