Homework #7

MATH 2030 Summer A 2021

Due: June 7, 2021

Problem 1

Systems of recurrence relations generalize recurrence relations the same way that systems of ODEs generalize ODEs. A (2-dimensional) recurrence relation is a system of equations of the form.

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} F_a(a_{n-1}, b_{n-1}) \\ F_b(a_{n-1}, b_{n-1}) \end{pmatrix}$$

A homogenous linear recurrence relation is one of the form $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = M \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix}$ for a 2×2 matrix A.

i) Consider the second order recurrence relation

$$a_n = a_{n-1} + a_{n-2}.$$

By setting $\dot{a}_n = a_{n-1}$, find a matrix M so that we can write this as an equivalent linear system

$$\begin{pmatrix} a_n \\ \dot{a}_n \end{pmatrix} = M \begin{pmatrix} a_{n-1} \\ \dot{a}_{n-1} \end{pmatrix}.$$

ii) Given arbitrary a_0, b_0 show that $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = M^n \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$ is the general solution to a homogeneous linear recurrence relation.

iii) Further show that if λ_1 and λ_2 are eigenvalues of M with eigenvectors $\vec{\xi}_1$ and $\vec{\xi}_2$ then the general solution can be written as

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = c_1 \lambda_1^n \vec{\xi}_1 + c_2 \lambda_2^n \vec{\xi}_2.$$

Hint: write $\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = c_1 \vec{\xi}_1 + c_2 \vec{\xi}_2$ and use part ii) and the fact that $M^n \vec{\xi}_1 = \lambda_1^n \vec{\xi}_1$.

iv) For the matrix M in part i), show that the eigenvalues of M are φ and $1 - \varphi$ where $\varphi = 1/2 + \sqrt{5}/2$ and that corresponding eigenvectors are $\begin{pmatrix} 1 \\ \varphi - 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -\varphi \end{pmatrix}$. Write the general solution to the equation using the formula in part iii) and find particular values of c_1 and c_2 to solve the "IVP" $a_1 = 1, \dot{a}_1 = a_0 = 0$.

v) This recurrence relation with initial data $a_0 = \dot{a}_1 = 0, a_1 = 1$ gives the Fibonacci sequence. Extract the first row of your solution to show that $a_n = F_n$ (the nth Fibonacci number) satisfies

$$F_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}(1-\phi)^n.$$

Problem 2

We will now use systems of recurrence relations to solve non-constant-coefficient systems of ODEs, or at least find approximations to their solutions for small values of t. Consider the system of ODEs

$$\vec{y}(t)' = \begin{pmatrix} \frac{1}{2} & 1-t^2 \\ 0 & \frac{1}{2} \end{pmatrix} \vec{y}$$

which is decoupled at $t = \pm 1$ but otherwise coupled.

i) Using the ansatz

$$\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} a_n t^n \\ \sum_{n=0}^{\infty} b_n t^n \end{pmatrix}$$

reduce the system of ODEs to a system of recurrence relations

$$\begin{aligned} a_{n+1} &= F_a(a_n, a_{n-1}, a_{n-2}, b_n, b_{n-1}, b_{n-2}) \\ b_{n+1} &= F_b(a_n, a_{n-1}, a_{n-2}, b_n, b_{n-1}, b_{n-2}). \end{aligned}$$

ii) For a general initial condition a_0, b_0 use the recurrence relation to find a_n and b_n for $n = 0, 1, 2, 3, 4$ and write the general solution as

$$\vec{y}(t) = \begin{pmatrix} a_0 + a_1 t + \cdots + a_4 t^4 + O(t^5) \\ b_0 + b_1 t + \cdots + b_4 t^4 + O(t^5) \end{pmatrix}.$$

Hint: $a_k = 0$ and $b_k = 0$ if $k < 0$.

Problem 3

For each of the following ODEs, find the characteristic exponents at each regular singular point and use this to write an ansatz for two independent solutions. You do not need to actually solve for the coefficients in the ansatz.

i) $x^2 y'' + 2xy' - 2e^x y = 0$.

ii) $x^2 y'' + \frac{1}{2}(x + \sin(x))y' + y = 0$. Hint: recall that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

Problem 4

Verify the following formulas for the Laplace transform:

i) Linearity
$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

ii) Frequency shift formula
$$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}(s-a).$$

Problem 5

Using the Laplace transform, solve the following IVP:

$$y'' + 2y' + 2y = h(t), \qquad y(0) = 0, y'(0) = 1$$

where
$$h(t) = \begin{cases} 1 & \pi < t \leq 2\pi \\ 0 & \text{otherwise} \end{cases}.$$

Problem 6

This is a bonus problem, and is not required. A complete solution gives 1 bonus point on this assignment for a total possible score of 11/10.

Consider the Bessel function of the first kind $J_\nu(x)$ for $\nu \in \mathbb{N}$ with power series representation

$$J_\nu(x) = \frac{x^\nu}{2^\nu \nu!} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\nu+1)_n 2^{2n}} x^{2n}$$

which is a solution of Bessel's equation. Show that

$$\mathcal{L}\{J_\nu(x)\} = \frac{(\sqrt{s^2+1}-s)^\nu}{\sqrt{s^2+1}}.$$

Hint: there are two methods. 1. Apply the Laplace transform to the Bessel equation, subtract the $Y(s)$ term from both sides and the left hand side is exact, so we can solve the equation. The solution will be the unique one whose inverse Laplace transform is bounded at 0. 2. Use linearity of the Laplace transform to take the transform of the power series for $J_\nu(x)$ term-by-term. Then match this with the Taylor expansion of the proposed solution around $\sigma = 0$ where $\sigma = 1/s$.