INTEGRABLE SYSTEMS

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1. What is a spin chain?

In quantum mechanics, every electron has a *spin*, which lives in the spin 1/2 representation of $SU(2)$.

This is a vector $\vec{v} \in \mathbb{C}^2$, equipped with a natural action of three matrices, $\sigma_x, \sigma_y, \sigma_z$, which have commutation relations

$$[\sigma_i, \sigma_j] = \epsilon^{ijk}\sigma_k$$

where the ϵ denotes the totally antisymmetric tensor on three indices.

All the matrices σ_i have eigenvalues ± 1. We view the eigenstates of σ_z as being of 'definite spin up in the z-plane', and 'definite spin down in the z-plane'. Of course, a general vector will not be of definite spin in any direction. But our geometrical picture of states as living in the x-y-z plane is justified, because normalised states in $\mathbb{C}^2$ form $\mathbb{CP}^1$, which is isomorphic to S^2.

Spin is important because, for instance, an electron acquires energy dependent on its spin in a magnetic field B,

$$Energy \propto \vec{\sigma} \cdot \vec{B}$$

When we say *spin chain*, what we mean is a collection of electrons along a 1D line which

- Cannot move in space;
- Have spin states $state \in \bigotimes_{spins} \mathbb{C}^2$ evolving in time according to an 'interaction Hamiltonian', a matrix in $End(\bigotimes_{spins} \mathbb{C}^2)$, by the rule

$$state \to e^{-iHt} state$$

- Interact at 'short range', typically modelled by only interacting with their nearest neighbours

Example. XXZ model.

Fix a chain of N spins in a line,

$$States = \otimes_{i=1}^{N} \mathbb{C}^2$$

Then define an operator

$$H_{XXZ} : States \to States$$

$$H_{XXZ} = -\sum_{n=1}^{N} \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z$$

Here, when we write $\sigma_n^x \sigma_{n+1}^x$, we mean the operator $1 \otimes \dots 1 \otimes \sigma_x \otimes \sigma_x \otimes 1 \otimes \cdots \otimes 1$. That all of these terms are nontrivial only along two adjacent spins satisfies the short-range condition above.

We want to solve the eigenvalue problem for this operator, and see how it depends on the parameter Δ. Here σ_i^j is the j-th Pauli matrix operating on the i-th tensor factor, where $\sigma_{N+1}^j = \sigma_1^j$ should the chain be periodic. Eventually, we want to understand this problem in the limit $N \to \infty$.

2. Basic observations

First, observe that $\sum_i \sigma_i^z$ commutes with the Hamiltonian, since

$$\sum_{j,k,i} [\sigma_i^j, \sigma_i^k \sigma_{i+1}^k] + [\sigma_i^j, \sigma_{i-1}^k \sigma_i^k] = \sum_{j,k,i,\ell} \epsilon^{jk\ell} \sigma_i^\ell \sigma_{i+1}^k + \epsilon^{jk\ell} \sigma_{i-1}^k \sigma_i^\ell$$
$$= \sum_{ijk\ell} \epsilon^{jk\ell} (\sigma_i^\ell \sigma_{i+1}^k + \sigma_i^k \sigma_{i+1}^\ell)$$

A symmetric term contracted with the Levi-Civita tensor is zero.

If $\Delta = 1$, the Hamiltonian H_{XXZ} has an 'enhanced symmetry' group $SL(2, \mathbb{C})$ since we can rotate along the Bloch sphere x, y, z without changing the Hamiltonian. Similar story at $\Delta = -1$.

Therefore, we expect roughly that there should be different behaviour in the regions

- $\Delta < -1$
- $-1 < \Delta < 1$
- $1 < \Delta$

With very special, 'critical' behaviour at the points $\Delta = \pm 1$.

Deep into the $\Delta < -1$ phase, for instance, we may assume $|\Delta| >> 1$ and then

$$H_{XXZ}/|\Delta| \sim \sum_i \sigma_i^z \sigma_{i+1}^z$$

so we get something Ising-like. Similar for $\Delta > 1$. But what we really want to focus on is the middle region where the dynamics are the most interesting,

$$-1 < \Delta < 1$$

3. Our game-plan

We will solve this H_{XXZ} by the following rough means: we will find a function whose value is a rank-four tensor $R(\lambda)$ with nice properties so that

$$\frac{d}{d\lambda} R_{i,i+1}(\lambda)_{\lambda=0} = H_{i,i+1}$$

Then nice properties of $R(\lambda)$ will imply algebraic structure about H_{XXZ} and its solutions. We will find

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{sinh(2i\eta)}{sinh(\lambda+2i\eta)} & \frac{sinh(\lambda)}{sinh(\lambda+2i\eta)} & 0 \\ 0 & \frac{sinh(\lambda)}{sinh(\lambda+2i\eta)} & \frac{sinh(2i\eta)}{sinh(\lambda+2i\eta)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

here written as a 4×4 matrix for readability rather than the more natural $2 \times 2 \times 2 \times 2$ rank-four tensor. Here $\Delta = \cos(2\eta)$. I imagine this matrix looks quite mysterious, and we'll talk about how one could write it down, but first we'll talk about what we can do with it.

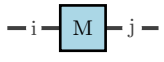
From R, we will find a related matrix $t(\lambda)$ so that $\partial_\lambda log(t(\lambda)) = H$, and so that $[t(\lambda), t(\nu)] = 0$ for all λ, ν. Then since the family t is diagonalisable, we can differentiate the eigenvalue equation and find

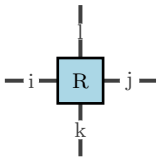
$$t(\lambda) |v\rangle = \tau(\lambda) |v\rangle$$
$$\implies H(\lambda) |v\rangle = \frac{\tau'(\lambda)}{\tau(\lambda)} |v\rangle$$

So, solving the eigenvalue problem for the family t would solve the eigenvalue problem for the Hamiltonian. This strategy might make it sound like we are just making things very complicated for ourselves, but we'll find that this gameplan is very general and effective.

Physically, this matrix R will come from a classical statistical model related to H_{XXZ}, but I will omit discussing this for conciseness, and instead just pull R from thin air.

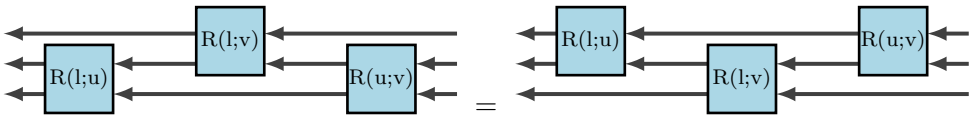
4. Algebraic structure of R

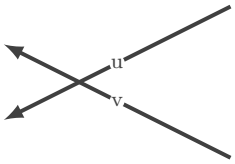
4.1. **Graphical setup.** We can view a matrix M as a rank-two tensor graphically via the following notation: 



Similarly, we can draw a rank-four tensor R like

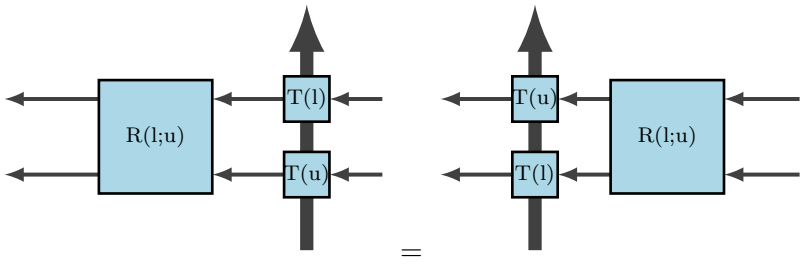
We will require that the tensor R discussed previously satisfies the **Yang-Baxter equation**, which means the equality of the two following pictures:





We imagine that $R(u, v)$ swaps the order of u, v arrows:
In this sense, the Yang-Baxter equation declares that all ways to swap v, u, l to l, u, v are equivalent.

Given such a rank-four tensor R, we associate an algebra of tensors T satisfying the **RTT relation**:



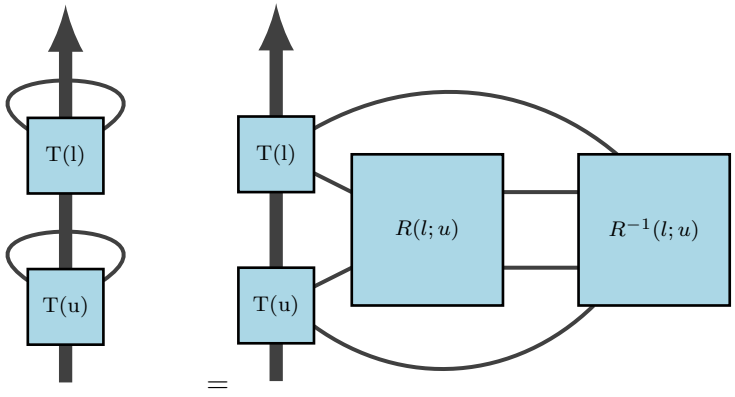
Here the thick line upwards denotes an auxiliary tensorial index.

Example. Fix a constant parameter κ_0. Then $R(u, \kappa_0)$, twisted so that its upper-left and bottom-right indices are treated as auxiliary, satisfies the RTT relation, by the Yang-Baxter relation.

In fact, we can chain copies of R together in strings this way and pull them through one-by-one, so chains like this also are tensors satisfying the RTT relation.

Definition. Given a tensor $T(u)$ so associated in this fashion, we let $t(u)$ denote the matrix formed by tracing over all physical indices. Then the

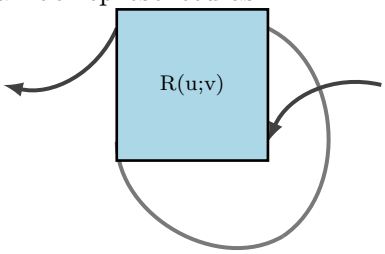
equality



followed by passing through the tensors T implies that

$$[t(u), t(l)] = 0$$

So, all the $t(u)$ are simultaneously diagonalisable, as we hoped for earlier. Note that explicitly, in terms of the R we have been working with, the matrix t can be represented as



4.2. **Solving for eigenstates.** Writing

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

as a matrix of matrices in the basis of physical states, and assuming the existence of a state

$$A(\lambda)\,|\Omega\rangle = a(\lambda)\,|\Omega\rangle$$
$$D(\lambda)\,|\Omega\rangle = d(\lambda)\,|\Omega\rangle$$
$$C(\lambda)\,|\Omega\rangle = 0$$

We want to produce eigenstates of $t(\lambda) = A(\lambda) + D(\lambda)$.

The way we try to do is is by treating B like a raising operator. We try

$$B(\lambda)\,|\Omega\rangle$$

And see what happens when we act by A, D on it. Our key tool to find relations will be the RTT relations, which will tell us

$$\langle i|R_{12}(\mu,\nu)T_1(\mu)T_2(\nu)|j\rangle = \langle i|T_1(\nu)T_2(\mu)R_{12}(\mu,\nu)|j\rangle$$

Where we take matrix elements with respect to the physical indices. With an eye to future generalisation, assuming only that

$$R(\mu,\nu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\mu,\nu) & c(\mu,\nu) & 0 \\ 0 & c(\mu,\nu) & b(\mu,\nu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To be very explicit, set $i = 0, 0 = j$. We find

$$\begin{aligned} &\langle 00|RTT|00\rangle \\ &= \langle 00|R_{12}(\mu,\nu)|00\rangle \, \langle 00|T_1(\mu)T_2(\nu)|00\rangle \\ &= A(\mu)A(\nu) \end{aligned}$$

whilst the other way around, we get $A(\nu)A(\mu)$.

We find relations like

$$\begin{aligned} B(\mu)B(\nu) &= B(\nu)B(\mu) \\ A(\mu)B(\nu) &= \frac{B(\nu)A(\mu) - b(\nu,\mu)B(\mu)A(\nu)}{c(\nu,\mu)} \\ D(\mu)B(\nu) &= \frac{B(\nu)D(\mu) - b(\mu,\nu)B(\mu)D(\nu)}{c(\mu,\nu)} \end{aligned}$$

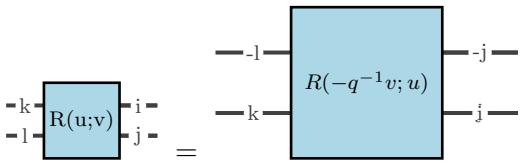
Therefore,

$$\begin{aligned} A(\mu)B(\nu)\,|\Omega\rangle &= \frac{1}{c(\nu,\mu)}[B(\nu)A(\mu)\,|\Omega\rangle - b(\nu,\mu)B(\mu)A(\nu)\,|\Omega\rangle] \\ &= \frac{a(\mu)}{c(\nu,\mu)}B(\nu)\,|\Omega\rangle - \frac{a(\nu)b(\nu,\mu)}{c(\nu,\mu)}B(\mu)\,|\Omega\rangle \\ D(\mu)B(\nu)\,|\Omega\rangle &= \frac{d(\mu)}{c(\mu,\nu)}B(\nu)\,|\Omega\rangle - \frac{d(\nu)b(\mu,\nu)}{c(\mu,\nu)}B(\mu)\,|\Omega\rangle \\ \implies t(\mu)B(\nu)\,|\Omega\rangle &= [\frac{a(\mu)}{c(\nu,\mu)} + \frac{d(\mu)}{c(\mu,\nu)}]B(\nu)\,|\Omega\rangle - [\frac{a(\nu)b(\nu,\mu)}{c(\nu,\mu)} + \frac{d(\nu)b(\mu,\nu)}{c(\mu,\nu)}]B(\mu)\,|\Omega\rangle \end{aligned}$$

Under the further assumption that

$$\frac{b(\nu,\mu)}{c(\nu,\mu)} = -\frac{b(\mu,\nu)}{c(\mu,\nu)}$$

True in our case, a reflection of the more abstract property of crossing symmetry



where above $-j$ denotes the value $NOT(j)$, and $\Delta = \frac{q+q^{-1}}{2}$

We find that the second term cancels if and only if

$$a(\nu) = d(\nu)$$

For a *fixed* ν. We find in general that

Claim.

$$B(\nu_1)\dots B(\nu_n)\,|\Omega\rangle$$

is an eigenstate of $t(\mu)$ if and only if for all $j \in 1,\dots,n$

$$\frac{d(\nu_j)}{a(\nu_j)} = \prod_{i=1, i\neq j}^{n} \frac{c(\mu_j,\mu_i)}{c(\mu_i,\mu_j)}$$

In this case, the eigenvalue of $t(\mu)$ is

$$a(\mu)\prod_{j=1}^{n}\frac{1}{c(\nu_j,\mu)} + d(\mu)\prod_{j=1}^{n}\frac{1}{c(\mu,\nu_j)}$$

This is what we mean by the *Bethe ansatz*: starting from a 'pseudovacuum', as above, we can rapidly produce very many eigenstates, with an explicit formula for their eigenvalues.

So for instance in our XXZ model discussion above, consider the state

$$|\Omega\rangle = |\uparrow\rangle \otimes \cdots \otimes |\uparrow\rangle$$

Where $\sigma_z\,|\uparrow\rangle = |\uparrow\rangle$ is the state of definite spin in the up-z direction. Then writing out matrices, we have on individual spins

$$\begin{aligned}
A(\mu) &\iff T_0^0(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & c(\mu) \end{pmatrix} \\
B(\mu) &\iff T_0^1 = \begin{pmatrix} 0 & 0 \\ b(\mu) & 0 \end{pmatrix} \\
C(\mu) &\iff T_0^1 = \begin{pmatrix} 0 & b(\mu) \\ 0 & 0 \end{pmatrix} \\
D(\mu) &\iff T_1^1 = \begin{pmatrix} c(\mu) & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Here the $\iff$ means 'corresponds to', roughly, but of course e.g. if we build our states out of two particles, we have $B(\mu) = \sum T_i^0 T_1^i$.

To be explicit about this, what we're doing to get e.g. is evaluating R at physical in-index 0 and out-index 0. So A has matrix elements $A_{ij} = \langle j0|A|0i\rangle$; i.e. $A_{00} = 1$, $A_{01} = 0$, $A_{10} = 0$, $A_{11}(\mu) = \langle 10|A|01\rangle = c(\mu,\kappa_0)$.

When we write $c(\mu)$, we are explicitly suppressing the second entry κ_0 on which c depends. It turns out that in our case $c(\mu,\nu) = c(\mu-\nu)$ and it makes sense to choose $\kappa_0 = 0$.

Hence,

$$\begin{aligned}
A(\mu)\,|\Omega\rangle &= |\Omega\rangle \\
D(\mu)\,|\Omega\rangle &= c(\mu)^L\,|\Omega\rangle \\
C(\mu)\,|\Omega\rangle &= 0
\end{aligned}$$

As desired. Notice that the operator B creates down-spins. L here is the length of the chain.

Explicitly substituting in the a, b, c for the XXZ spin chain, we find equations (where L is the length of the chain)

$$(\frac{cosh(\nu_j - i\pi/2)}{cosh(\nu_j + 2i\eta - i\pi/2)})^L = \prod_{i=1, j\neq i}^{n} \frac{sinh(\nu_j - \nu_i - 2i\eta)}{sinh(\nu_j - \nu_i + 2i\eta)}$$

With energy

$$2\sum_{j=1}^{n} \frac{sin(2\eta)^2}{cos(2\eta) + cosh(2\nu_j + i\eta - i\pi/2)}$$

Having such a clean expression for the energy and wavefunction of a very general class of states is incredibly powerful. Of course, there are a few immediate problems. First, it is not clear from this description what the ground state (state of lowest energy) might be, though you can probably compute it easily enough. The second problem is that it is not clear at all whether you can get every eigenstate via this procedure. Third, it is not clear how we take the limit as the chain goes to infinity: the energy we have computed miraculously does not depend on L, nor does the algebraic form of our eigenstates, but the eigenvalue of $D(\mu)$ does, and so do the Bethe equations themselves. We will not really discuss these problems in further detail.

The idea, by the way, to prove that all eigenvectors are accounted for is to just count the number of distinct eigenstates you can get from the Bethe ansatz in the limit as L is large, and realise it's all the eigenstates.

5. Abstract symmetry of the Bethe ansatz equations

Instead, we will begin to take a representation-theoretic view on the problem. Rather than consider $T(z)$ as a function, we will consider it as a formal power series. But there is a wrinkle: the trigonometric functions in terms of which solutions are written have a lot of poles. It's not sufficient to take their expansion around zero only.

Definition. The quantum loop algebra for $\mathfrak{sl}_2$, $U_q\widehat{\mathfrak{sl}_2}$ is the algebra of operators occuring in the $z = 0$ and $z = \infty$ expansions of $T(z)$, for a tensor T solving the RTT relation above and the crossing relation, and one additional relation.

The fact that we need the $z = 0$ and $z = \infty$ expansions adds a little bit of complication to the story, and so we are going to move from the trigonometric case to the rational case by fixing $\Delta = 1$. Then, whilst $T(u)$ previously had a one-dimensional lattice of poles and so we could not only look at the $z = \infty$ expansion, it now has only one pole and we need only look at the $z = \infty$ coefficients.

We will henceforth move to the XXX spin chain, whose algebraic structure is governed by

Definition. The Yangian, $Y\mathfrak{gl}_2$, is the algebra of the expansion around $z = \infty$ of matrix elements of $T(z)$ subject to the RTT relation

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v)$$

Where $R = 1 - Pu^{-1}$ is the permutation matrix.

The Yangian $Y\mathfrak{sl}_2$ has the additional relation

$$1 = qdetT(u) = t_{22}(u-1)t_{11}(u) - t_{21}(u-1)t_{12}(u)$$

The coproduct sends $T_{ij}(u) \to \sum_k T_{ik}(u)T_{kj}(u)$

This motivates us to study how the Yangian transforms eigenstates of H_{XXX}. It turns out we will want **level one representations**, where only the order zero and one terms act nontrivially.

At this level, we can identify generators $1, \sigma_a^0, \sigma_a^1$, with coproduct

$$\Delta(1) = 1 \otimes 1$$
$$\Delta(\sigma_a^0) = \sigma_a^0 \otimes 1 + 1 \otimes \sigma_a^0$$
$$\Delta(\sigma_a^1) = \sigma_a^1 \otimes 1 + 1 \otimes \sigma_a^1 + \hbar\epsilon_{bc}^a \sigma^b \sigma^c$$

Here $\hbar$ can be thought of as just a formal degree-counting parameter.

We can act on states of H_{XXX} via taking the coproduct. We find

$$\Delta^{L-1}(\sigma_a^0) = \sum_{k=1}^{L} \sigma_k^a$$

apologies for the awful notation, and

$$\Delta^{L-1}(\sigma_a^1) = \sum_{k=1}^{L} \sigma_k^{a,1} + \hbar\epsilon_{bc}^a \sum_{k>l=1} J_k^b J_\ell^c$$

Setting $\sigma_k^{a,1}$ to act trivially, since H_{XXX} is manifestly symmetric under $\mathfrak{sl}_2$, the generators of the first type commute with the Hamiltonian obviously. *But* generators of the second type do not commute with the Hamiltonian; they telescope and leave boundary terms on the starting and ending sites.

So the Yangian is *not* a symmetry of H_{XXX} for finite spin chains, nor for periodic ones; only in the 'infinite' limit, where no boundary terms exist, can it be considered as a symmetry.