

Introduction to Hecke algebras and Affine Hecke algebras

1. Motivation (Hecke algebras in nature)

• Definition

A **Coxeter system** (W, S) is a group and a finite set $S \subset W$ such that $W = \langle S \mid R \rangle$, where the set of relations is:

- $s^2 = 1$ $\forall s \in S$ "quadratic"
- $\underbrace{stst \dots}_{m_{st}} = \underbrace{tst \dots t}_{m_{st}}$ $\forall s, t \in S$ "braid"

Example: $W = \text{Weyl group}$, $S = \text{simple reflections}$

The **Hecke algebra** associated to (W, S) is the unital associative algebra $H = H(W)$ over $\mathbb{Z}[v, v^{-1}]$ generated by the symbols $\{\delta_s : s \in S\}$ such that

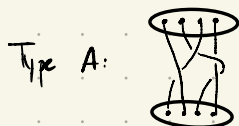
- $\delta_s^2 = (v^{-1} - v)\delta_s + 1$ "quadratic" $\leftarrow (\delta_s - v^{-1})(\delta_s + v) = 0$
- $\underbrace{\delta_s \delta_t \dots}_{m_{st}} = \underbrace{\delta_t \delta_s \dots}_{m_{st}}$ "braid"

Note that specializing to $v=1$ one gets the group algebra $\mathbb{C}[W]$

But why?

• Braid groups

The Braid group is the group generated by $\{T_s : s \in S\}$ subject only to the braid relations.



B_n
 \downarrow
 H
 \downarrow
 W

Representations of the braid group that factor through W have $\varphi(T_s)^2 = 1$.

Consider representations that satisfy the deformed $\varphi(T_s)^2 + p\varphi(T_s) + r$ $\xrightarrow{\text{no scale}}$ $\varphi(T_s)^2 = (q^{-1} - q)\varphi(T_s) + 1$

(many choices for a presentation, all isom.)

• Number theory

(G, K) $\Rightarrow H(G//K) = (K \backslash K)$ -invariant continuous functions $G \rightarrow \mathbb{C}$ of compact support.
unimodular, locally compact top group
 Algebra structure: convolution $(u, v) \mapsto \int_G u(g)v(g^{-1}x)dg$

Example: $G = GL_2(\mathbb{Q})$, $K = GL_2(\mathbb{Z})$ $\xrightarrow{\text{no}}$ $H(G//K) = \text{ring of Hecke operators on modular forms}$

(hence "Hecke", although it was Iwasawa who introduced them)

• Finite groups

Consider a finite group $G \geq B$, and an irrep ψ of H .

Now how does $\text{Ind}_B^G(\psi)$ decompose? "Unipotent principal series reps" $\text{End}_{\mathbb{C}G}(\text{Ind}_B^G \psi)$ will contain information (e.g., irrep $\leftrightarrow \dim=1$).

Clearly: $\{ \text{irreps in } \text{Ind}_B^G \psi \} \xleftrightarrow{1:1} \{ \text{irreps of } "H(G, B, \psi)" \}$ More precisely:

$$\text{End}_{\mathbb{C}G}(\text{Ind}_B^G \psi) = \text{Hom}_{\mathbb{C}G}(\text{Ind}_B^G \psi, \text{Ind}_B^G \psi) = \text{Hom}_{\mathbb{C}B}(\text{Res}_B^G \text{Ind}_B^G \psi, \psi) = \bigoplus_{B \backslash G / B} \text{Hom}_{\mathbb{C}B}(\text{Ind}_{B \cap gBg^{-1}}^B \text{Res}_{B \cap gBg^{-1}}^B \psi, \psi)$$

Setting $\psi = 1$, $\text{Ind}_B^G 1 = \bigoplus_{g \in B \backslash G / B} 1 \leftrightarrow (B \times 1)$ -invariant functions $G \rightarrow \mathbb{C}$

$\text{End}_{\mathbb{C}G}(\text{Ind}_B^G 1) \leftrightarrow (B \times B)$ -invariant functions $G \rightarrow \mathbb{C}$

Note that the algebra structure is again given by convolution: $(u * v)(x) = \frac{1}{|B|} \sum_{g \in G} u(g)v(g^{-1}x)$

Groups of Lie type: G (e.g. $\text{GL}_n(\mathbb{F}_q)$), B Borel mod $H(G, B, 1) = H_\psi(w) / (q = p^n)$

Another heuristic for why $H(\text{GL}_n(\mathbb{F}_q), B, 1)$ deforms S_n :

flags in $\mathbb{F}_q^n = \text{ordered basis in } \mathbb{F}_q^n = q^{\binom{n}{2}} \cdot \frac{q^n - 1}{q - 1} \cdot \frac{q^{n-1} - 1}{q - 1} \cdots \frac{q^2 - 1}{q - 1} = q^{\binom{n}{2}} (q^{n-1} + \dots + 1) (q^{n-2} + \dots + 1) \cdots (1)$

If $q=1$, this is $n!$, and since $\text{GL}_n(\mathbb{F}_q) \hookrightarrow \text{free} + \text{trans} \rightsquigarrow \text{GL}_n(\mathbb{F}_q) = S_n$, " $B = 1$ ", " $H(\text{GL}_n(\mathbb{F}_q), 1, 1) = S_n$ ".

• Quantum groups

Classical Schur-Weyl duality: $gl_n \subset V^{\otimes r} \supset S_r$ mod $\{ gl_n\text{-irreps in } V^{\otimes r} \} \xleftrightarrow{1:1} \{ S_r\text{-irreps in } V^{\otimes r} \} \xleftrightarrow{1:1} \text{partitions of } r$
(actions centralize each other)

Quantum Schur-Weyl duality: $U_q(gl_n) \subset V_q^{\otimes r} \supset H(S_n)$ mod r -dim
(q -deformed actions)

• Kazhdan-Lusztig theory

Two $\mathbb{Z}[v, v^{-1}]$ -bases:

• **Standard**: $\{ \delta_x : x \in W \}$ (Here we take a reduced expression $x = s_1 \cdots s_n$ and define $\delta_x = \delta_{s_1} \cdots \delta_{s_n}$)

• **Kazhdan-Lusztig**: $\{ b_x : x \in W \}$ characterized by: $\begin{cases} \bar{b}_x = b_x & (\text{KL involution}) \\ b_x = \delta_x + \sum_{y < x} h_{yx} \delta_y \end{cases}$ for some $h_{yx} \in v \mathbb{Z}[v]$ "degree bound"
 $\bar{\delta}_s := \delta_s^{-1} = \delta_s + (v - v^{-1})$ extended multiplicatively
 $\bar{v} := v^{-1}$

Let λ be dominant, $M(\lambda) := U(\mathfrak{m}) \otimes_{U(\mathfrak{g})} \mathbb{C}_\lambda$, $L(\lambda) := \text{simple module of h.w. } \lambda$. The KL conjecture says:

$$[M(y \cdot 0) : L(x \cdot 0)] = h_{yx} \Big|_{v=1}$$

• Link invariants

Traces on H mod Alexander, Jones polys

• **Categorical actions**: Hecke category $\subset \mathcal{O}$

• **Modular rep theory**: $H|_{q \rightarrow 1}$ (type A) decomposition matrices $\xrightarrow{\text{doubly indexed mod}} \text{decomp matrices of } S_n \text{ in char } p$

2. Representation theory of H for W finite.

What does H -mod look like? Spoiler: just like W -mod. However, we can specialize q to any element of \mathbb{C}^\times , and the representation categories will be different.

Let $z \in \mathbb{C}^\times$. We will denote $H_z := \mathbb{Z}[q^{\pm 1}] / (q - z) \otimes_{\mathbb{Z}[q^{\pm 1}]} H$.

First question: for what values of z is H_z ss?

Def (Trace form): If A is a f.d. K -algebra, denote $L_x: A \rightarrow A$ $a \mapsto xa$. Then the **trace form** is $(,): A \times A \rightarrow K$, $(x, y) = \text{Tr}(L_x L_y)$.

Remark: The trace can be defined for Hecke algebras even if they are not f.d.

Prop: A is ss $\Leftrightarrow (,)$ is nondegenerate

Proof:

\Rightarrow) Nondegeneracy can be checked by passing to \bar{K} . Now $A \otimes_{\bar{K}}$ is a product of matrix algebras over \bar{K} . These are simple and hence contain no double-sided ideals, in particular the radical of the form restricted to each is 0.

\Leftarrow) Recall A Artinian $\Rightarrow J(A) =$ largest nilpotent right ideal.

Now if $j \in J(A)$, ja is nilpotent for all $a \in A$. Pass to \bar{K} , upper triangularize the action of ja $\Rightarrow \text{Tr}(ja) = 0 \forall a \Rightarrow j \in \text{rad}((,))$.

Let $R = K[q^{\pm 1}]$. Assume A is an R -algebra, finite as an R -module. For $f \in K^\times$, denote $A_f = A \otimes_R R / (q - f)$.

Prop: If A_f is ss, then A is ss.

Proof: The discriminant of the trace form on A is $D(q)$, so if $D(q)|_{q=f} \neq 0$, $D(q) \neq 0$.

Cor: The generic Hecke algebra is semisimple.

Proof: The specialization to $q=1$ is $\mathbb{C}[W]$, which is semisimple.

Moreover, we have the following stronger result:

Theorem (Tits' Deformation thm): If H_1, H_2 are semisimple, then $H_1 \cong H_2$ abstractly.

Proof: By the previous proposition, the discriminant on H is nonzero, so $H \otimes \overline{\mathbb{C}(q)}$ is a product of matrix algebras over $\overline{\mathbb{C}(q)}$, of dimensions n_1, \dots, n_k , "the numerical invariants". It suffices to show the numerical invariants for H_1 are the same: n_1, \dots, n_k .

(so that $H_1 \cong H \otimes \overline{\mathbb{C}(q)}|_{q=1} \cong H \otimes \overline{\mathbb{C}(q)}|_{q=1} \cong H_1$) (omit)

Adjoin formal variables x_w for $w \in W$ and consider $H_{\overline{\mathbb{C}(q)}} \otimes \overline{\mathbb{C}(q)}(x_w: w \in W)$, in order to write a "generic element" $a = \sum x_w \delta_w$. Let $P(t)$ be its char poly, say $P(t) = \prod P_i(t)^{e_i}$ in $\overline{\mathbb{C}(q)}[t, x_w: w \in W]$ is the decomp into irreducibles.

Since $H_{\overline{\mathbb{C}(q)}} \otimes \overline{\mathbb{C}(q)}(x_w: w \in W) = \prod H_{n_i}(\overline{\mathbb{C}(q)}(x_w: w \in W))$, it has a basis $\{E_{ij}^i\}$ for each entry in each summand.

So write $a = \sum_{i,j} y_{ij}^i E_{ij}^i$ for $y_{ij}^i \in \overline{\mathbb{C}(q)}(x_w: w \in W)$. The change of basis matrix has entries in $\overline{\mathbb{C}(q)}$ so $\overline{\mathbb{C}(q)}(x_w: w \in W) = \overline{\mathbb{C}(q)}(y_{ij}^i)$. In this basis,

$$P(t) = \prod_i \det(t \cdot \text{id} - y_{ij}^i)^{n_i e_i}$$

Now specialize y_{ij}^i so that the $\det(t \cdot \text{id} - y_{ij}^i)$ are irred and distinct. Then $P_i(t) = \det(t \cdot \text{id} - y_{ij}^i)$ and $e_i = n_i = \deg P_i(t)$.

Now consider the generic element $a = \sum w \delta_w \in H_z \otimes \mathbb{C}(w: w \in W)$. By the same argument, since H_z is ss, its char poly, $P_e(t)|_{q=z}$, has its irreducible factors appearing with multiplicity = degree. Since $n_e = \deg P_e(t)|_{q=z}$, the $P_e(t)|_{q=z}$ must be irred and distinct. Hence the n_e are the numerical invariants for H_z too. \square

Conclusion: H_z for generic z is isomorphic to $\mathbb{C}[W]$.

When exactly? Whenever $z \notin \{D(q) = 0\}$. This amounts to: H_z is ss iff $z^{2/(n-1)} \sum_{w \in W} z^{2/(w)} \neq 0$.

For type A, this amounts to: H_z is ss iff $\text{order}(z) > n$ or if $n \geq 3$, $z=0$ also works.

3. Affine Hecke algebras and their representations

Reference for 2nd half: MIT-Northeastern 2017 DAHA/HA seminar notes.

Definition: not quite $H(W, S)$ for W affine, but close.

Motivation:

- Reductive p -adic groups

Example: $G = GL_n(\mathbb{Q}_p)$, $I \subseteq GL_n(\mathbb{Z}_p) \Rightarrow C_c(I \backslash G / I) = H(W_{aff}, s = p)$. "Iwahori spherical algebra"
Iwahori subgroup

- K -theory

G complex ss simply connected Lie group, $N = \text{nilpotent cone}$, $\bar{N} \rightarrow N$ Springer resolution, $Z = \bar{N} \times_{\bar{N}} \bar{N}$ "Steinberg variety"

Then $K^b(Z) = \mathbb{Z}[W_{aff}]$ and $K^{G \times \mathbb{C}^\times}(Z) = H_{aff}$

- First step to understand Cherednik algebras

Immediate problem: Tits' deformation argument **fails**. Rep theory of specializations becomes much more involved.

(For the rest I use a single reference: MacDonal, AHAs and orthogonal polynomials)

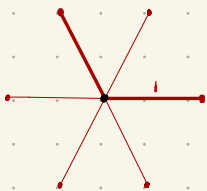
Affine stuff

Fix a finite irreducible reduced system $R \subset V$. Here V has an inner product (\cdot, \cdot) . Embed $V \subset V \oplus \mathbb{C}\delta$, with $\delta \perp V$.

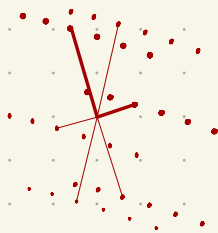
Then the associated **affine root system** is $R^a := \{\alpha + n\delta : n \in \mathbb{Z}\} \subset F$

$$\delta : V \rightarrow \mathbb{C} \\ x \mapsto 1$$

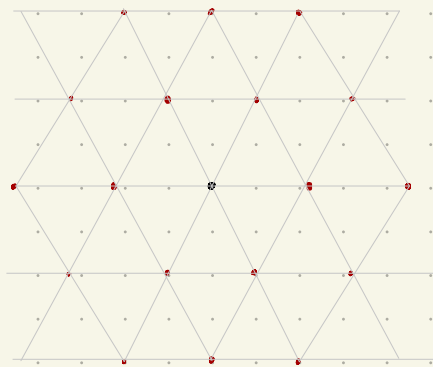
Writing $\alpha^v = \frac{2\alpha}{(\alpha, \alpha)}$, we have $Q := \sum_{\alpha \in R} \mathbb{Z}\alpha$ **root lattice** $Q^v := \sum_{\alpha^v \in R^v} \mathbb{Z}\alpha^v$ **coroot lattice**



R



R^a



Q

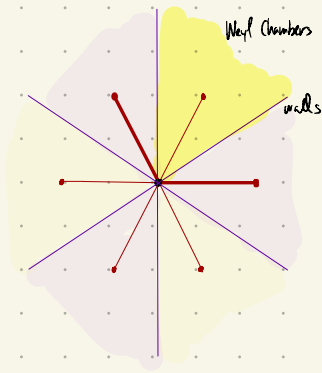
Then $W = \langle s_\alpha : \alpha \in R \rangle$, $W^a = \langle s_\alpha : \alpha \in R^a \rangle$

For $v \in V$, denote $t(v) : V \rightarrow V$. Then $t(v)(\alpha) = \alpha - (v, \alpha)\delta$

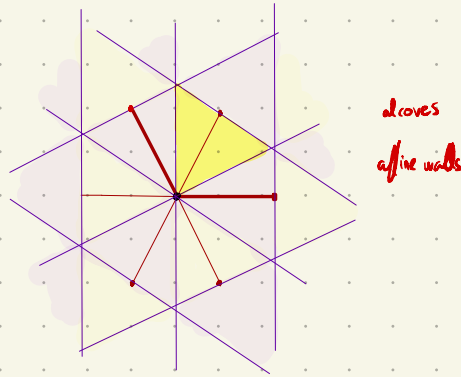
$$x \mapsto x + v$$

Then $t(Q^v) \cong W^a$ and $W^a = W \rtimes t(Q^v)$. "affine Weyl group"

Zero focus of R :



Zero focus of R^\vee :

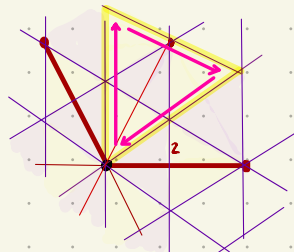


Consider the "fundamental alcove". This is an n -simplex with $n+1$ walls given by affine walls corresponding to some $a_0, a_1, \dots, a_n \in R^\vee$. These are the **simple affine roots**. Note $\alpha_i = a_i$. Fact: if $\theta \in R^+$ is the highest root, $a_0 = -\theta + \delta$.

$$A = \langle s_i = s_i, s_0 = s_0 \rangle$$

Notice that $W^{ae} := W \ltimes t(P^\vee)$ (dominant coweights) also acts on R^\vee . $t(\lambda)(a) = a - \underbrace{(\lambda, a)}_{\in \mathbb{Z}} \delta$.

This admits a length function extending that of W^* and W . Note that $\Omega := \{w \in W^{ae} : l(w) = 0\} = \{w \in W^{ae} : wA = A\}$ is a finite group.



$$\Omega = C_3 \quad \begin{array}{c} a_0 \\ \triangle \\ a_1 \quad a_2 \end{array}$$

We have $\Omega = P^\vee/Q^\vee$ (Dynkin diagram automorphisms) and $W^{ae} = \Omega \ltimes W^*$. Note that $\Omega \subset A$ via $\Omega \subset$ affine simple roots, so if $\pi_r(a_i) = a_j$, $\pi_r s_i \pi_r^{-1} = s_j$, hence the semidirect product.

Braid groups

The **Braid group** of a Coxeter system is the group generated by $\{T_w : w \in W\}$, subject only to the braid relation.

The braid relation is equivalent to: $T_w T_{w'} = T_{ww'}$ whenever $l(ww') = l(w) + l(w')$.

Define the **affine braid group** B^a as that of (W^a, I) , and the **extended affine braid group** B^{ae} as that of (W^{ae}, I) (with its length function).

B^{ae} has two important subgroups.

• The elements T_π with $\pi \in \Omega$ form a subgroup of B^{ae} isomorphic to Ω (obvs), and $B^{ae} = \Omega \ltimes B^a$, where if $\pi_r(a_i) = a_j$, $\pi_r T_i \pi_r^{-1} = T_j$.

• For $\lambda \in P_+^\vee$, define $Y^\lambda = T_{\epsilon(\lambda)}$, for $\mu, \nu \in P^\vee$, define $Y^{\mu+\nu} = Y^\mu (Y^\nu)^{-1}$. These generate a copy of P^\vee .

Proposition: $T_1, \dots, T_n, Y^\lambda: \lambda \in P^\vee$ generate B^{ae} as a group (notice the absence of T_0)

$$\begin{aligned} \text{If } (\lambda, \alpha_i) = 0 & \text{ then } T_i Y^\lambda = Y^\lambda T_i \\ (\lambda, \alpha_i) = 1 & \text{ then } T_i Y^\lambda = Y^\lambda T_i^{-s_i \lambda} \quad (Y^\lambda = T_i Y^{s_i \lambda} T_i) \end{aligned}$$

(idea: reduce to $\lambda \in D_+^\vee$ and use properties of the length function)

The previous proposition leads to a presentation of B^{ae} reminiscent of $W^{ae} = W \ltimes \mathbb{Z}(P^\vee)$:

$$B^{ae} = \langle T_1, \dots, T_n, Y^{P^\vee} \mid \begin{array}{l} T_i Y^\lambda = Y^\lambda T_i \quad (\alpha_i, \lambda) = 0 \\ T_i Y^\lambda = Y^\lambda T_i^{-s_i \lambda} \quad (\alpha_i, \lambda) = 1 \end{array} \rangle$$

We can finally state:

Definition (AHA): The **affine Hecke algebra** $H(W^{ae})$ is the quotient of the group algebra of B^{ae} by the Hecke relations: $(T_i - q)(T_i + q^{-1}) = 0$

How do the T_i and Y^λ interact in $H(W^{ae})$?

Lemma: $T_i Y^\lambda - Y^{s_i(\lambda)} T_i = (q - q^{-1}) \frac{Y^{s_i \lambda} - Y^\lambda}{Y^{-\alpha_i^\vee} - 1}$

Proof: A calculation shows that if this holds for Y^λ and Y^μ , then it holds for $Y^{-\lambda}$ and $Y^{\lambda+\mu}$. So two cases to check:

- $(\lambda, \alpha_i) = 0 \Rightarrow$ this says $T_i Y^\lambda = Y^\lambda T_i$
- $(\lambda, \alpha_i) = 1 \Rightarrow$ this says $T_i Y^\lambda - Y^{s_i(\lambda)} T_i = (q - q^{-1}) Y^\lambda$

"(Prop)"

$$T_i Y^\lambda - T_i^{-1} Y^\lambda = T_i Y^\lambda - (T_i^{-1} + q - q^{-1}) Y^\lambda \text{ as desired } \square$$

We have given two presentations of B^{ae} : Coxeter ($B^{ae} = \Omega \ltimes B^a$) and Bernstein ($B^{ae} = \langle T_i, Y^\lambda | \dots \rangle$). This implies the following.

Prop: $H(W^{ae}) \cong \Omega \ltimes H(W^a)$, therefore $\{T_w : w \in W^{ae}\}$ is a \mathbb{C} -basis for $H(W^{ae})$.

- The subalgebra generated by T_i (including 0) is isom to $H(W^a, S)$
- T_i (not including 0) $H(W, S)$

Q: Basis for the second presentation?

Fact: as \mathbb{C} -v.s., $H(W, S) \otimes_{\mathbb{C}} \mathbb{C}Y^{P^\vee} \xrightarrow{\sim} H(W^{ae})$, so $\{T_w Y^\lambda : w \in W, \lambda \in P^\vee\}$ is another basis.

This map allows us to construct many representations of $H(W^{ae})$: for E a rep of $H(W, S)$, $\text{Ind} E := H(W^{ae}) \otimes_{H(W)} E$

As a $\mathbb{C}Y^{P^\vee}$ -module, $\text{Ind} E = \mathbb{C}Y^{P^\vee} \otimes E$

In particular, if $E = \mathbb{C}$ by specializing $q = z$, we get $\mathbb{C}Y^{P^\vee}$ "Polynomial representation"

Now the last lemma implies that T_i acts by $z s_i + (z - z^{-1}) \frac{s_i - 1}{Y^{\alpha_i^\vee} - 1}$ in $\mathbb{C}Y^{P^\vee}$

Remarks: • One can modify this action to $\beta: T_i \mapsto z s_i + (z - z^{-1}) \frac{s_i - 1}{Y^{\alpha_i^\vee} - 1}$ now acting on $\mathbb{C}[X]$ (group algebra of the weight lattice)

This is called Cherednik's basic representation.

- Both of these representations are faithful.
- In fact DAHAs can be defined as the 2-parameters (q, z) subalgebra of $\text{End}_{\mathbb{C}}(\mathbb{C}[X])$ gen by $X^{\lambda} \beta(T_w)$ ($\lambda \in P, w \in W^{ae}$).

We finish by computing the center of the AHA.

Theorem: $Z(H(W^{ac})) = (\mathbb{C}Y^{P^v})^W$.

Proof: \supseteq Easy: if $f = f(Y) \in (\mathbb{C}Y^{P^v})^W$, $T_i f - f T_i = 0$ (since $T_i Y^\lambda - Y^{s_i(\lambda)} T_i = (q - q^{-1}) \frac{Y^{s_i \lambda} - Y^\lambda}{Y^{-\alpha_i} - 1}$ by our lemma).
Since $\mathbb{C}Y^{P^v}$ is commutative and the T_i and the Y^λ generate, $f \in Z(H(W^{ac}))$.

\subseteq By the lemma, if $f \in \mathbb{C}Y^{P^v}$, $T_i f - s_i f T_i = g(Y) \in \mathbb{C}Y^{P^v}$.

If f is central, $(f - s_i f) T_i = g(Y) \xRightarrow{\text{basis}} f = s_i f$.

So it suffices to see that $f \in \mathbb{C}Y^{P^v}$. Now under the polynomial representation $\mathbb{C}Y^{P^v}$, f commutes with all Laurent polynomials, hence its image is a Laurent polynomial. But the polynomial representation is faithful so $f \in \mathbb{C}Y^{P^v}$. \square