RATIONAL CHEREDNIK ALGEBRAS

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Today, I am going to explain how a large class of interesting problems are related to the rational degeneration of DAHA. Most of these problems have nondegenerate analogues, which are more complicated to state but still very interesting. I hope introducing all these small interesting classes of problems in a fairly classical setting will help ground us going forwards in the seminar.

1. A classical example

Consider the rational Calogero-Moser Hamiltonian for two particles with coordinates x and y interacting on a line with an inverse-squared potential,

$$H = \partial_x^2 + \partial_y^2 - c(c+1)\frac{1}{(x-y)^2}$$

It is easy to explicitly show this Hamiltonian is completely integrable.
Write

$$D_x(g) = \partial_x + \frac{g}{x-y}(1 - P_{xy})$$
$$D_y(g) = \partial_y + \frac{g}{y-x}(1 - P_{xy})$$

Claim. D_x, D_y commute.

Proof. By explicit computation. □

Acting on functions symmetric in x and y,

$$H_1 = D_x(g) + D_y(g) = \partial_x + \partial_y$$
$$H_2 = D_x(g)^2 + D_y(g)^2 = \partial_X^2 + \partial_y^2 + 2g\frac{\partial_x - \partial_y}{x-y}$$

...because the operator $(1 - P_{xy})$ vanishes on such functions.

Claim. H_1, H_2 commute.

Proof. By expanding their definitions and using that the D_x, D_y commute. □

Now let

$$\tilde{H}_1 = (x-y)^g H_1 (x-y)^{-g}$$
$$\tilde{H}_2 = (x-y)^g H_2 (x-y)^{-g}$$

We can compute

$$\tilde{H}_2 = \partial_x^2 + \partial_y^2 - 2g(g-1)\frac{1}{(x-y)^2}$$

is our original Calogero-Moser Hamiltonian.

Corollary 1. The two-particle Calogero-Moser Hamiltonian is completely integrable (which just means that there is an operator which commutes with the Hamiltonian).

Remark. One may show that the spectral problem

$$H_1 J(x,y) = (m_x + m_y)J(x,y)$$
$$H_2 J(x,y) = (m_x^2 + m_y^2)J(x,y)$$

Has solutions

$$J^{(g)}_{m_x m_y}(x,y) = e^{(m_x+m_y)(x+y)/2}((m_x - m_y)(x-y))^{1/2-g} I_{g-1/2}((m_x - m_y)(x-y)/2)$$

Where $I_\nu(z)$ is the modified Bessel function of the first kind. Therefore, understanding eigenfunctions of the Calogero-Moser system is a generalisation of the classical theory of Bessel functions.

If you are not familiar with Bessel functions, the way they most classically arise is the following: attempt to solve Laplace's equation in three dimensions, by separation of variables in spherical coordinates. Then the radial solutions, up to some prefactors, are Bessel functions. This is a shadow of a more general relation involving symmetric spaces.

2. General Calogero-Moser Hamiltonians

Definition. Let V a finite dimensional complex vector space. A **complex reflection** $s \in GL(V)$ is a semisimple element conjugate to the diagonal matrix $diag(\lambda_s, 1, \dots, 1)$ for $\lambda \neq 1$.

If V has an inner product, a **real reflection** $s \in O(V)$ is a semisimple element conjugate to $diag(-1, 1, \dots, 1)$.

In what follows, I'd like to note that pretty much every statement "works" for complex reflections. But I will only discuss the case of real reflections.

Many formulas which are given in terms of just $c(s)$ in the real reflection case will be given in terms of $\frac{2c(s)}{1-\lambda_s}$ in the complex case. They reduce since $\lambda_s = -1$ in the real reflection case.

Setup. Let $\mathfrak{h}$ be a finite dimensional complex vector space with inner product $(\bullet, \bullet)$. Let $W \subset O(\mathfrak{h})$ a finite group generated by a set of real reflections $S \subset W$.

W acts on S by conjugation; let $c : S \to \mathbb{C}$ be a conjugation-invariant function.

Let $\alpha_s \in \mathfrak{h}^\star$ be an eigenvector so $s\alpha_s = -\alpha_s$.

Definition. The quantum Olshanetsky-Perelomov Hamiltonian is

$$H = \Delta - \sum_{s \in S} \frac{c(s)(c(s)+1)(\alpha_s, \alpha_s)}{\alpha_s^2}$$

I will now explain the tools needed to show complete integrability of the above Hamiltonian.

Definition. Let $a \in \mathfrak{h}$. Define the **Dunkl operator** to be

$$D_a(c) := \partial_a - \sum_{s \in S} \frac{c(s)\alpha_s(a)}{\alpha_s}(1-s)$$

Claim. The Dunkl operators commute amongst themselves.

Definition. If B is a differential operator, let $m(B)$ be the differential operator restricted to act on W-invariant functions.

Definition. The algebra $\mathbb{C}[\mathfrak{h}]^W$ is free. Let $P_0, P_1, \dots, P_r$ be its homogeneous generators.

Pick an orthonormal basis y_i of $\mathfrak{h}$.

Let $L_i = m(P_i(D_{y_i}, \dots, D_{y_r}))$.

Claim. $L_1 = \Delta - \sum_s \frac{c_s(\alpha_s, \alpha_s)}{\alpha_s} \partial_{\alpha_s^\vee}$.

Further, Let $\delta(c)(x) = \prod_s \alpha_s(x)^{c(s)}$. Then

$$H_{O-P} = \delta(c)^{-1} \circ L_1 \circ \delta(c)$$

Corollary 2. The Olshanetsky-Perelomov quantum system is completely integrable.

Remark. Let's explain how to reduce this to our classical example. Consider $S_2 \subset GL(\mathbb{C}^2)$, acting by permuting coordinates. Then there is a unique generating reflection, P_{xy}. The equation $P_{xy}\alpha = -\alpha$ is solved by the vector $(1, -1)$. So for instance,

$$D_x(c) = \partial_x - \sum_s \frac{c\alpha(e_1)}{\alpha}(1 - P_{xy}) = \partial_x - \frac{c}{x-y}(1 - P_{xy})$$

The algebra $\mathbb{C}[x,y]^{S_2} = \mathbb{C}[x+y, x^2+y^2]$. Therefore $P_0 = x+y, P_1 = x^2+y^2$.

Remark. This is not the unique A_1 example of a Dunkl operator. If we represent A_1 as $\mathbb{Z}/2\mathbb{Z}$ acting on just $\mathbb{C}$, with $sf(x) = f(-x)$, there is a unique Dunkl operator,

$$D_x = \partial_x - \frac{c}{x}(1-s)$$

So it's really important to work in the generality of real reflection groups, and not just root systems.

3. The Rational Cherednik algebra

We have proposed an understanding of the Olshanetsky-Perelomov Hamiltonian in terms of Dunkl operators.

If we want to understand the common eigenfunctions of the L_i we have defined, a good start is to understand how the operators L_i act on polynomial functions and power series, which can be reduced to understanding the commutators of Dunkl operators.

Claim. Fix $x \in \mathfrak{h}^\star$. Then

$$[D_a, x] = (a, x) - \sum_{s \in S} c(s)(a, \alpha_s)(x, \alpha_s^\vee)s$$

Definition. We can view $c : S \to \mathbb{C}$ as an element in $\mathbb{C}W$. Using this viewpoint, let *Dunkl* be the subalgebra in $\mathbb{C}W \ltimes DifferentialOperators(\mathfrak{h}_{reg})$ generated by the Dunkl operators D_a and coordinates $x \in \mathfrak{h}^\star$.

They satisfy the relations

$$\begin{aligned} [x, x'] &= 0 \\ [D_a, D_{a'}] &= 0 \\ [D_a, x] &= (a, x) - \sum_{s \in S} c(s)(a, \alpha_s)(x, \alpha_s^\vee)s \end{aligned}$$

Here $\mathfrak{h}_{reg}$ refers to the subset so $\alpha_s(x) \neq 0$ – i.e. the subspace on which all the Dunkl operators are nonsingular.

This algebra has a natural filtration inherited from $DifferentialOperators(\mathfrak{h}_{reg})$, by order of differential operators. We can take the associated Rees algebra, $A = \oplus_{n=0}^{\infty} F^n A$, of direct sums of filtered pieces.

Claim. The Rees algebra of *Dunkl* is the quotient of $\mathbb{C}W \ltimes Tensor(T^\star \mathfrak{h})[\hbar]$ by the relations

$$\begin{aligned} [x, x'] &= 0 \,\forall x, x' \in \mathfrak{h}^\star \\ [y, y'] &= 0 \, y, y' \in \mathfrak{h} \\ [y, x] &= \hbar(y, x) - \sum_{s \in S} c(s)(y, \alpha_s)(x, \alpha_s^\vee)s \end{aligned}$$

Call this H_c.

Proof. Since the graded version of $D_a(c)$ is $\hbar(a, \bullet) - \sum_s \frac{c(s)\alpha_s(a)}{\alpha_s}(1-s)$. Working out the commutation relations, what changes is only the term coming from ∂_a. ☐

Definition. Call H_c the **rational Cherednik algebra**, or the **degenerate DAHA**.

Example. In type A_1, corresponding to $\mathfrak{h} = \mathbb{C}, W = \mathbb{Z}/2\mathbb{Z}$, we have the algebra generated by s, x, y, with relations

$$\begin{aligned} s^2 &= 1 \\ sx &= -xs \\ sy &= -ys \\ [y, x] &= t - 2cs \end{aligned}$$

Where here we have set $t = \hbar(x, y)$.

Remark. It is useful to write down a basis for the rational Cherednik algebra. Let y_i be a basis of $\mathfrak{h}$, and x_i be the dual basis of $\mathfrak{h}^\star$. Then the elements y_i, x_i, s_i form a basis. Note that the elements s_i could be indexed by a different set from the y_i, x_i. In terms of these elements,

$$\begin{aligned} [y_i, x_j] &= \hbar\delta_{ij} - \sum_{s\in S} c_s(y_i, \alpha_s)(x, \alpha_s^\vee)s \\ &= \hbar\delta_{ij} - \sum_{s\in S} \frac{2c_s(y_i, \alpha_s(y_j, \alpha_s)}{(\alpha_s, \alpha_s)} s \end{aligned}$$

Example. Write $W = S_n$, acting on $\mathfrak{h} = \mathbb{C}^n$. Then there is only one reflection-invariant function $c : S^n \to \mathbb{C}$, so c is just a number. An eigenvector α_{ij} of the transposition s_{ij} is $y_i - y_j$.

The only nonzero $\alpha_{k\ell}$ with nonzero dot product with both y_i, y_j for $i \neq j$ is α_{ij} itself. Hence

$$[y_i, x_j] = \frac{2c}{2} s_{ij} = cs_{ij}$$

Likewise,

$$[y_i, x_i] = t - c \sum_{j\neq i} \frac{2(y_i, \alpha_{ij})(y_i, \alpha_{ij})}{2} s_{ij} = t - c \sum_{j\neq i} s_{ij}$$

Note that signs cancel since $(-1)^2 = 1$.

Further study of rational DAHA is very interesting, and proceeds largely via the theory of $\mathcal{D}$-modules. One may sheafify the rational Cherednik algebra over $\mathfrak{h}_{reg}/W$, basically declaring that $H_c(U)$ has sections polynomials and Dunkl operators over U.

The study of rational DAHA basically also proceeds unchanged for complex reflection groups, as mentioned above.

4. Solving the O-P Hamiltonian

Suppose now that we could solve the equation

$$x_\alpha \Psi = D_\alpha \Psi$$

This equation depends on operators s_α and x_α. Therefore, in the most general setting, Ψ is a function taking values in some representation of the algebra generated by the s_α, x_α. This algebra is the degenerate affine Hecke algebra.

Recall. The **degenerate affine Hecke algebra** is the associative algebra generated by $\mathbb{C}W$ and $x_1, \dots, x_n$ with relations

$$[x_i, x_j] = 0$$
$$[s_i, x_j] = 0, i \neq j$$
$$s_i x_i = k + x_i s_i - \sum_{j=1}^{n} (\alpha_i^\vee, \alpha_j) x_j s_i$$

So, suppose for some function Ψ valued in a representation V of the degenerate affine Hecke algebra, we have an equality

$$x_\alpha \Psi = D_\alpha \Psi$$

That would imply

$$p(x_1, \dots, x_n)\Psi = p(D_1, \dots, D_n)\Psi$$

Since Dunkl operators commute. If we knew also that

(1) $p(x_1, \dots, x_n) = p(\lambda_1, \dots, \lambda_n)$ in V, for and reflection-invariant polynomial and some fixed $\vec{\lambda}$;
(2) There exists a linear map $tr : V \to \mathbb{C}$ which is reflection-invariant

We could solve the O-P Hamiltonian as follows. By property 1), $P_i(\lambda_1, \dots, \lambda_n)\Psi = L_i\Psi$. These are representation-valued functions: taking tr, we get

$$P_i(\lambda_1, \dots, \lambda_n) tr(\Psi) = L_i(\Psi)$$

To be a solution of the Olshanetsky-Perelomov Hamiltonian. We conclude

Claim. In the above setting, the map tr defines a map from solutions of the system $x_\alpha \Psi = D_\alpha \Psi$ to eigenfunctions of the Olshanetsky-Perelomov Hamiltonian.

Remark. The equation $x_\alpha \Psi = D_\alpha \Psi$ is *almost* the KZ equation after conjugation by $\delta(c)$. See the next set of lecture notes for a precise relation to the KZ equation.

Why have we defined Dunkl operators so that we need to conjugate them to get H_{O-P}? The reason is that, in the form we've chosen for Dunkl operators, they have direct q-analogues, whilst in the conjugate form they do not.

Remark. The map defined above is actually an isomorphism, for a large class of modules V_λ. The easiest way to show this is to show a more general result in the setting of the trigonometric degeneration, where the monodromy can be more explicitly calculated.

5. Basics of the representation theory of the rational Cherednik algebra

Hopefully the above was motivating to consider the representation theory of the rational Cherednik algebra. Henceforth set $\hbar = 1$.

Remark. Just like DAHA, the rational Cherednik algebra has a symmetrising element

$$e = |W|^{-1} \sum_{w \in W} w$$

The nonunital subalgebra

$$eH_c e$$

is called the **spherical subalgebra** of H_c.

Notice that the Olshanetsky-Perelomov hamiltonians live in the spherical subalgebra, for instance.

Claim. The elements

$$h = \frac{1}{2} \sum_i \{x_i, y_i\}$$
$$E = -\frac{1}{2} \sum_i x_i^2$$
$$F = \frac{1}{2} \sum_i y_i^2$$

Form an $\mathfrak{sl}_2$-triple.

Hence, there is a natural grading with raising/lowering operators on any module M.

Proof. For once, let's do a computation. Observe we have

$$[h.x_j] = \sum_i [x_i y_i + y_i x_i, x_j] = \sum_i x_i [y_i, x_j] + [y_i, x_j] x_i$$
$$= \frac{1}{2} \sum_i x_i (\delta_{ij} + \sum \frac{2(y_i, \alpha_s)(y_j, \alpha_s)}{(\alpha_s, \alpha_s)} s) + (\delta_{ij} + \sum \frac{2(y_i, \alpha_s)(y_j, \alpha_s)}{(\alpha_s, \alpha_s)} s) x_i$$

The two second terms cancel since all s in the sum anticommute with x_i. So We are left with

$$\frac{1}{2} \sum_i 2 x_i \delta_{ij} = x_j$$

So we conclude

$$[h, x_j] = x_j$$

Therefore,

$$[h, E] = -\frac{1}{2}\sum_j [h, x_j^2] = -\frac{1}{2}\sum_j x_j[h, x_j] + [h, x_j]x_j = -\frac{1}{2}\sum_j 2x_j^2 = -E$$

□

This $\mathfrak{sl}_2$-triple inside the rational Cherednik algebra is a relic of the $SL(2,\mathbb{Z})$-symmetry of DAHA that Henry mentioned last time.

Remark. The rational Cherednik algebra also has a PBW decomposition, given by an isomorphism

$$\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^\star][\hbar] \to gr(H_c)$$

Our PBW theorem implies a triangular decomposition $H_c = S\mathfrak{h}^\star \otimes \mathbb{C}W \otimes S\mathfrak{h}$. Whenever we have a triangular decomposition like this, we can define an associated category $\mathcal{O}$, and totally formally from such a triangular decomposition one can prove a lot of properties for $\mathcal{O}$.

Definition. The category $\mathcal{O}_c(W, \mathfrak{h})$ is the category of modules over the rational Cherednik algebra which are finitely generated over $S\mathfrak{h}^\star$ and are locally finite under $S\mathfrak{h}$.

Claim. There is a decomposition $\mathcal{O} = \oplus_{\lambda \in \mathfrak{h}^\star/W}\mathcal{O}_\lambda$, where $\mathcal{O}_\lambda$ is the subcategory of modules M so

$$\forall v \in M, \forall p \in S\mathfrak{h}^W, \exists N : (p - \lambda(p))^N v = 0$$

Here $\mathfrak{h}^\star/W = Specm(\mathfrak{S}h)^W$.

Proof. By local finiteness under $S\mathfrak{h}$, and checking that for $x \in \mathfrak{h}^\star$,

$$(p - \lambda(p))^{N+1}xv = (N+1)(\partial_x p)(p - \lambda(p))^N v$$

□

6. How is this algebra a degeneration?

Recall that DAHA contains elements $Y^{\nu^\vee}$, for all $\nu^\vee$ the cocharacter lattice $Hom(\mathbb{C}^\times, T)$.

The **rational degeneration** amounts to replacing the cocharacter lattice $Hom(\mathbb{C}^\times, T)$ with $Hom(\mathbb{C}^\times, Lie(T))$, and also the character lattice $Hom(T, \mathbb{C}^\times)$ with $Hom(Lie(T), \mathbb{C}^\times)$.

Basically, this means we should differentiate all relations.

But we need to do it in a slightly careful way. Nonetheless, let me be a little sketchy in the following example, for speed.

Example. Consider A_1 DAHA. It has relations

$$TXT = X^{-1}$$
$$T^{-1}YT^{-1} = Y^{-1}$$
$$Y^{-1}X^{-1}YXT^2 = q$$
$$(T-t)(T+t^{-1}) = 0$$

In the first step, we differentiate only along the cocharacter lattice. So we set X to be constant, set $Y_{|0} = id, (\delta Y)_{|0} = y$, and differentiate T only in relations containing Y, setting $T_{|0} = s, (\delta T)_{|0} = c, t = 0, \delta q_{|0} = 1, q_{|0} = 0$.

Our relations now read

$$sXs = X^{-1}$$
$$s^2 = 1$$
$$-cs + sys - cs = \delta(T^{-1}YT^{-1}) = \delta(Y^{-1}) = -y$$
$$\implies sy + ys = 2c$$
$$-ys^2 + X^{-1}yX = 1 - 2cs$$

The algebra generated by these elements is called the **trigonometric degeneration** of A_1 DAHA, alternatively the **trigonometric Cherednik algebra of type** A_1. A further differentiation in X results in the rational Cherednik algebra of type A_1.