PARABOLIC INDUCTION AND RESTRICTION

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Today, we are going to discuss category $\mathcal{O}$ for the rational Cherednik algebra.

Recall. The **rational Cherednik algebra** associated to a finite group W generated by reflections acting on a vector space $\mathfrak{h}$ and a conjugation invariant function $W \to \mathbb{C}$, c is the quotient of $\mathbb{C}W \ltimes Tensor(\mathfrak{h} \oplus \mathfrak{h}^\star)$ by relations

$$[x, x'] = 0$$
$$[y, y'] = 0$$
$$[y, x] = (y, x) - \sum_s c_s(y, \alpha_s)(x, \alpha_s^\vee)s$$

Here x is a vector and y is a covector. We denote this algebra $H_c(W, \mathfrak{h})$.

Example. Our standing example today will be the case where $c = 0$. Then the relations specialise to

$$[x, x'] = 0$$
$$[y, y'] = 0$$
$$[y, x] = (y, x)$$

These are relations of the Weyl algebra of $\mathfrak{h}$, equivalently the algebra of differential operators on $\mathfrak{h}$, $\mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$.

So

$$H_0(W, \mathfrak{h}) = \mathbb{C}W \ltimes Weyl(\mathfrak{h})$$

Therefore, study of the rational Cherednik algebra on a vector space generalises the study of differential operators on that vector space.

0.1. **The triangular decomposition.** To start with, we have a PBW theorem.

Claim. Filter $H_c(W, \mathfrak{h})$ by setting $deg(y) = 1, deg(x) = 0 = deg(s_\alpha)$. The natural surjective map

$$\xi : \mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h} \otimes \mathfrak{h}^\star][\hbar] \to gr(H_c)$$

is an isomorphism. Hence, we have a triangular decomposition

$$H_c = \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^\star]$$

Very generally, given *any* such triangular decomposition, we can define a category $\mathcal{O}$ and prove basic properties about it.

Definition. The category $\mathcal{O}_c(W,\mathfrak{h})$ is the category of $H_c(W,\mathfrak{h})$-modules finitely generated over $\mathbb{C}[\mathfrak{h}]$ and locally finite under $\mathbb{C}[\mathfrak{h}^\star]$.

Definition. This category has **standard modules**, $M_c(W,\mathfrak{h},\tau) := Ind_{W\otimes\mathbb{C}[\mathfrak{h}^\star]}^{H_c(W,\mathfrak{h})}(\tau)$, where τ is a finite-dimensional W-representation equipped with the zero action of $\mathbb{C}[\mathfrak{h}^\star]$.

Here are some totally formal consequences of our definitions. See the paper [GGOR] for explanations.

Claim. There is a generalised weight space decomposition

$$\mathcal{O}_c(W,\mathfrak{h}) = \oplus_{\lambda\in\mathfrak{h}^\star/W}\mathcal{O}_c(W,\mathfrak{h})_\lambda$$

Where

$$M_\lambda = \{m \in M | p \in \mathbb{C}[\mathfrak{h}^\star]^W \implies (p-\lambda(p))^n = 0, n >> 0\}$$

Claim. Let M be an $H_c(W,\mathfrak{h})$-module. Then, the following are equivalent:

(1) $M \in \mathcal{O}_c(W,\mathfrak{h})_0$, i.e. $\mathbb{C}[\mathfrak{h}]^\star$ acts locally nilpotently;
(2) M is a quotient of a sum of standard modules.
(3) M has an ascending filtration whose succesive quotients are quotients of standard modules.

Further, any standard module $M_c(\tau)$ has a maximal proper submodule $J_c(\tau)$. Denote $L_c(\tau) = M_c(\tau)/J_c(\tau)$ to be the irreducible module quotient.

Corollary 1. The simple objects of $\mathcal{O}_c(W,\mathfrak{h})_0$ are the $L_c(\tau)$.

0.2. **The grading element.**

Claim. Let

$$h = \frac{1}{2}\sum_i \{x_i, y_i\}$$
$$E = -\frac{1}{2}\sum_i x_i^2$$
$$F = \frac{1}{2}\sum_i y_i^2$$

Then, h, E, F form an $\mathfrak{sl}_2$-triple. We call h the **grading element**

Claim. h acts on $S^m\mathfrak{h}^\star \otimes \tau$ by the scalar element $h_c(\tau) + m$, where

$$h_c(\tau) = \frac{dim\,\mathfrak{h}}{2} - \sum_{s\in S} c_s s_{|\tau}$$

Example. Consider $H_0(W,\mathfrak{h})$ again. Then

$$h = \frac{1}{2}\sum_i x_i\partial_i + \partial_i x_i = \frac{dim(\mathfrak{h})}{2} + \sum_i x_i\partial_i$$
$$E = -\frac{1}{2}\sum_i x_i^2$$
$$F = \frac{1}{2}\sum_i \partial_i^2$$

We can easily compute, for instance, that

$$[h,E] = \frac{1}{2}\sum_{i,j}[x_i\partial_i, x_j^2] = \frac{1}{2}\sum_{i,j} x_i[\partial_i, x_j^2] = \sum_{i,j}\delta_{ij}x_ix_j = 2E$$

The statement that h acts on $\tau \otimes S^m\mathfrak{h}^\star$ by a fixed scalar here just means that for any homogeneous degree m polynomial, we have

$$\sum_i x_i\partial_i p = mp$$

Claim. If M is finitely generated over $\mathbb{C}[\mathfrak{h}]$, then $M \in \mathcal{O}_c(W,\mathfrak{h})_0$ if and only if h acts locally finitely on M.

Proof. If $M \in \mathcal{O}_c(W,\mathfrak{h})_0$, then $\mathbb{C}[\mathfrak{h}^\star]$ acts locally nilpotently on M. So for $m \in M$, $\mathbb{C}[\mathfrak{h}^\star]m$ is a finite dimensional vector space.

There is a nonempty subvector space U of vectors u so $yu = 0$, for all $y \in \mathfrak{h} \subset S\mathfrak{h} \simeq \mathbb{C}[\mathfrak{h}^\star]$.

On U, view

$$h = \frac{1}{2}\sum_i \{x_i, y_i\} = \sum_i x_iy_i + \frac{1}{2}[y_i, x_i]$$

The first term vanishes on U, and the second (commutator) term is in $\mathbb{C}W$. Therefore, h acts locally finitely on U (meaning powers of h span only a finite dimensional vector space).

Now apply induction, assuming we know the result on the lesser dimensional space V/U to conclude for all of V.

Conversely, we can grade $M = \oplus_{\beta\in B}M[\beta]$ by generalised eigenvalues of h. Since M is finitely generated over $\mathbb{C}[\mathfrak{h}]$, commutation relations imply B is a finite union of sets $z_i + \mathbb{Z}_{\geq 0}$. On each of these, $\mathbb{C}[\mathfrak{h}]$ acts locally nilpotently. □

Corollary 2. Any finite dimensional $H_c(W,\mathfrak{h})$-module is in $\mathcal{O}_c(W,\mathfrak{h})_0$. Further, any module in $\mathcal{O}_c(W,\mathfrak{h})_0$ has a grading by generalised eigenvalues of h.

Corollary 3. Grade $H_c(W,\mathfrak{h})$ by assigning all the x_i degree one and all the y_i degree zero. Then, there is a fully faithful functor

$$\mathcal{O}_c(W,\mathfrak{h})_0 \to H_c(W,\mathfrak{h}) - mod^{\mathbb{C}-gr}$$

Further, we have a decomposition

$$H_c(W,\mathfrak{h}) - mod^{\mathbb{C}-gr} = \oplus_{[\alpha]\in\mathbb{C}/\mathbb{Z}} H_c(W,\mathfrak{h}) - mod^{\alpha+\mathbb{Z}-gr}$$

Corollary 4. If $h_c(\tau) - h_c(\tau')$ is never a positive integer, over all τ, τ' irreps of W, then $\mathcal{O}_c(W,\mathfrak{h})_0$ is semisimple, and its simple objects are the $M_c(\tau)$.

Proof. If $N \subset M_c(\tau)$ is a proper submodule, then it has weight vectors integrally less than $h_c(\tau)$. So it can only admit a nontrivial map from a standard module with $h_c(\tau') \in h_c(\tau) - \mathbb{N}$. But every proper submodule admits a map from some $M_c(\tau')$, so if no such map exists, the proper submodule is trivial.

If $h_c(\tau) - h_c(\tau') \notin \mathbb{Z}$, $Ext^i(M_c(\tau), M_c(\tau'))$ vanish for $i \geq 0$ by the fully faithful embedding and decomposition of the prior corollary.

Of course, it could be that $h_c(\tau) = h_c(\tau')$. Then the extension

$$0 \to M_c(\tau') \to N \to M_c(\tau) \to 0$$

must be split by a map $M_c(\tau) \to N$, so $Ext^1(M_c(\tau), M_c(\tau')) = 0$. □

0.3. Generic semisimplicity.

Claim. A vector is **singular** if $\mathfrak{h} \subset S\mathfrak{h}$ annihilates it. A module is singualar if every vector inside it is singular.

If U is an $H_c(W,\mathfrak{h})$-module, and $S \subset U$ is a $\mathbb{C}W$-submodule of singular vectors, there is a unique map $\phi : M_c S \to U$ of $\mathbb{C}[\mathfrak{h}]$ modules so that $\phi_{|S}$ is the identity. This map is an H_c-homomorphism.

Proof. Because $M_c(S)$ is a free module over $\mathbb{C}[\mathfrak{h}]$ generated by S.

□

Corollary 5. Let $K = max_\tau h_c(\tau)$, where τ runs over all irreps. Then for any $M \subset N \in \mathcal{O}_c(W,\mathfrak{h})_0$, if $M[\beta] = N[\beta]$ for all $Re(\beta) \leq K$, then $M = N$.

Proof. Then M/N begins in degree $Re(\beta) > K$. But if M/N is nonzero, it must admit a nontrivial map from some $M_c(\tau)$. Therefore must start in degree at most $h_c(\tau)$. □

Corollary 6. Any module $M \in \mathcal{O}_c(W,\mathfrak{h})_0$ has finite length.

Proof. Any module has a bounded below starting h-generalised eigenvalue, so the bound above implies that there can be only a finitely long quotient sequence.

□

Example. Remember our $c = 0$ example. Here $h_c(\tau)$ is independent of τ, and always equal to $\frac{dim(\mathfrak{h})}{2}$. Therefore, the above theorem tells us that the $\mathcal{O}_0(W,\mathfrak{h})$ is semisimple, with simple objects isomorphic to $M_0(\tau)$.

The map

$$Rep(W) \to \mathcal{O}_0(W,\mathfrak{h})$$
$$\tau \to M_0(\tau)$$

is a map of semisimple categories sending simple objects to simple objects, therefore it is an equivalence of categories.

Claim. Fix any c so that $h_c(\tau) - h_c(\tau')$ is never a positive integer. This is a generic condition. Then

$$\mathcal{O}_c(W, \mathfrak{h}) \simeq Rep(W)^{f.d.}$$

1. Regular induction and restriction functors

There are classical induction and restriction functors.

Definition. Let $H \subset G$. There is a functor

$$res : Rep(G) \to Rep(H)$$

Which sends a G-representation $G \to GL(V)$ to the induced map $H \subset G \to GL(V)$.

Definition. There is also an induction functor

$$ind : Rep(H) \to Rep(G)$$

Which, in the finite group case we are in, sends a representation $\pi \in \mathbb{C}[H] - mod$ to the module

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \pi \in \mathbb{C}[G] - mod$$

2. Parabolic induction and restriction

Let $W_\lambda \subset W$ be the isotropy group of $\lambda \in \mathfrak{h}^\star$. Then for generic c, meaning c so $h_c(\tau) - h_c(\tau') \notin \mathbb{Z}$, we can define induction and restriction functors in a lazy way by using the isomorphisms we have. So we define, for instance,

$$res_\lambda : \mathcal{O}_c(W, \mathfrak{h}) \simeq Rep(W) \to Rep(W_\lambda) \simeq \mathcal{O}_{c_{|W_\lambda}}(W^\lambda, \mathfrak{h}/\mathfrak{h}^{W_\lambda})$$

Our goal is to define these functors for *all* c, and relate them to the KZ connection. In addition, here we make a lot of arbitrary choices because lots of choices we can make lead to isomorphic categories. These choices will become important over nongeneric c, where the categories involved are not going to be isomorphic to $Rep(W)$. It is worth giving an example.

Example. Consider $H_c(\mathbb{Z}/2\mathbb{Z}, \mathbb{C})$. Irreducible representations of $\mathbb{Z}/2\mathbb{Z}$ are the trivial representation and the sign represenation. Therefore

$$h_{trivial}(\tau) = \frac{1}{2} - c$$
$$h_{sign}(\tau) = \frac{1}{2} + c$$

And $h_{sign}(\tau) - h_{trivial}(\tau) = 2c$. Therefore, when c is an integer or a half-integer, it's possible that the $M_c(\tau)$ are not simple.

The module $M_c(sign)$ is $\mathbb{C} \otimes \mathbb{C}[x]$ as a vector space, by PBW.

We have

$$yx = xy + 1 - 2cs$$
$$sx = -xs$$

This implies, for instance, that

$$\begin{aligned} yx^3 = xyx^2 + (1-2cs)x^2 &= x^2yx + x(1-2cs)x + (1-2cs)x^2 \\ &= x^2(1-2cs) + x(1-2cs)x + (1-2cs)x^2 \\ &= x^2 + 2cx^2 + x^2 - 2cx^2 + x^2 + 2cx^2 \\ &= 3x^2 + 2cx^2 \end{aligned}$$

Therefore, if $c = -\frac{3}{2}$, $< x^3 >$ is a proper submodule. The irreducible quotient $L_{-3/2}(sign) = \mathbb{C}[x]/x^3$.

Remark. This is all related to the classical fact that the Weyl algebra doesn't have any finite dimensional representations. For concreteness, let's look at $\mathbb{C}[x, \partial]$. If there was a finite dimensional representation π, then

$$[\pi(\partial), \pi(x)] = \pi[\partial, x] = \pi(1) = 1$$

A well-defined trace map $tr : V \to \mathbb{C}$ would then imply that $dim(V) = 0$, since the trace of a commutator must be zero.

Another advantage of our geometric approach is that it will become clear that we can define a functor

$$Res : \mathcal{O}_c(W, \mathfrak{h})_0 \to \mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'}) \boxtimes Loc({\mathfrak{h}^{\star}_{reg}}^{W'})$$

Whose fibre over $\lambda \in Loc({\mathfrak{h}^{\star}_{reg}}^{W'})$ is the functor Res_λ. Then, using that these functors vary in a 'flat' way, we'll find

$$Res(M_c(\tau)) = \oplus_i M_c(W', \mathfrak{h}/\mathfrak{h}^{W'}, \tau_i) \otimes \mathcal{L}_{\tau,\tau_i}$$

Where $\mathcal{L}_{\tau,\tau_i}$ is a local system given by the trivial bundle with connection

$$\nabla = d - \sum_{s \in S, s \notin W'} \frac{c_s d\alpha_s}{\alpha_s}(1-s)$$

Which is, up to conjugacy, a relative version of the KZ connection.

3. A geometric POV

To extend our induction and restriction functors to be defined for all c, we need to view $H_c(W, \mathfrak{h})$ as a sheaf of modules over $\mathfrak{h}/W$, by defining for an affine open $U \hookrightarrow \mathfrak{h}/W$,

$$H_c(W, \mathfrak{h})(U) := \mathbb{C}[U] \otimes_{\mathbb{C}[\mathfrak{h}]^W} H_c(W, \mathfrak{h})$$

Claim. Alternately, $H_c(W, \mathfrak{h})(U)$ can be described as the algebra of operators on $\mathcal{O}(U \times_{\mathfrak{h}/W} \mathfrak{h})$ generated by $\mathcal{O}(U \times_{\mathfrak{h}/W} \mathfrak{h})$, W, and Dunkl operators.

Definition. Let $\widehat{H}_c(W, \mathfrak{h})_b$ denote the completion at $b \in \mathfrak{h}/W$.

Let $\widehat{\mathcal{O}_c}(W, \mathfrak{h})$ denote the category of $\widehat{H}_c(W, \mathfrak{h})_0$-modules finitely generated over $\mathbb{C}[[\mathfrak{h}]]$.

There is a completion functor

$$\widehat{\bullet} : \mathcal{O}_c(W,\mathfrak{h}) \to \widehat{\mathcal{O}_c(W,\mathfrak{h})}$$

$$M \to \widehat{M} := \widehat{H_c(W,\mathfrak{h})}_0 \otimes_{H_c(W,\mathfrak{h})} = \mathbb{C}[[\mathfrak{h}]] \otimes_{\mathbb{C}[\mathfrak{h}]} M$$

The latter equality arising by the Dunkl description.

Definition. Let

$$E : \widehat{\mathcal{O}_c(W,\mathfrak{h})} \to \mathcal{O}_c(W,\mathfrak{h})_0$$

Be the map sending a module N to the span of generalised eigenvectors of h in N. Note that any such generalised eigenvector must have a finite power series, as we'll demonstrate shortly, and h acts locally finitely, so we indeed land in $\mathcal{O}_c(W,\mathfrak{h})_0$.

Claim. The restriction of $\widehat{\bullet}$ to $\mathcal{O}_c(W,\mathfrak{h})_0 \subset \mathcal{O}_c(W,\mathfrak{h})$ is an equivalence of categories, whose inverse is given by E.

Proof. $M \subset \widehat{M}$, and therefore $M \subset E(\widehat{M})$. To show the opposite inclusion, pick generators $\{m_i\}$ of M over $\mathbb{C}[\mathfrak{h}]$ which are generalised eigenvectors of h.

Any $v \in E(\widehat{M})$ can be written as $v = \sum_i f_i m_i$, where the $f_i \in \mathbb{C}[[\mathfrak{h}]]$.

Remember now that $hx_i = x_i(h+1)$. Hence,

$$(h-\mu)v = \sum_{d,i} f_i^{(d)}(d+h-\mu)m_i$$

So if $(h-\mu)^N v = 0$, then

$$\sum_{d,i} f_i^{(d)}(d+\mu_i-\mu)^N m_i = 0$$

We conclude that $d = \mu - \mu_i$. In particular, $f_i = f_i^{(\mu-\mu_i)}$ is a polynomial.

We also need to show that $\widehat{E(N)} = N$, true if the natural map $E(N) \to N/\mathfrak{m}^j N$ is surjective for every j. h preserves the descending filtration on N by $\mathfrak{m}^j N$, so $gr(N)$ admits a locally finite action of h, with finite dimensional eigenspaces.

So $N/\mathfrak{m}^{j+1}N[\beta] \to N/\mathfrak{m}^j N[\beta]$, for fixed β, is surjective for all j and must be an isomorphism for large enough j. □

Remark. The same proposition holds, using $E_\lambda : \widehat{\mathcal{O}_c(W,\mathfrak{h})} \to \mathcal{O}_c(W,\mathfrak{h})_\lambda$.

Definition. The **generalised Jacquet functor** $J_\lambda : \mathcal{O}_c(W,\mathfrak{h}) \to \mathcal{O}_c(W,\mathfrak{h})_\lambda$ is the composition of the completion functor with E_λ. The restriction $J_{\lambda|\mathcal{O}_c(W,\mathfrak{h})_\lambda}$ is the identity functor.

Claim. There is a natural isomorphism

$$\theta : \widehat{H}_c(W,\mathfrak{h})_b \to End_{\widehat{H}_{c'}(W^b,\mathfrak{h})_0}(Fun_{W^b}(W, \widehat{H}_{c'}(W^b,\mathfrak{h})_0))$$

Defined by

$$\theta(u)(f)(w) = f(wu),\ u \in W$$
$$\theta(x_\alpha)(f)(w) = (x^{(b)}_{w\alpha} + (w\alpha, b))f(w)$$
$$\theta(y_{\alpha^\vee})(f)(w) = y^{(b)}_{w\alpha^\vee} f(w) + \sum_{s \in S, s \notin W_b} \frac{c_s \alpha_s w(\alpha^\vee)}{x^{(b)}_{\alpha_s} + \alpha_s(b)}(f(sw) - f(w))$$

By natural Morita equivalence for matrix algebras, this descends to a functor

$$\theta_\star : \mathcal{O}_c(W, \mathfrak{h})_\lambda \to \mathcal{O}_{c'}(W^\lambda, \mathfrak{h})_0$$

Here c' denotes the restriction of the function c to the parabolic subgroup W^λ.

Since W^λ acts trivially on $\mathfrak{h}^{W_\lambda}$, this extends to an equivalence of categories

$$\psi_\lambda : \mathcal{O}_c(W, \mathfrak{h})_\lambda \to \mathcal{O}_{c'}(W^\lambda, \mathfrak{h}/\mathfrak{h}^{W^\lambda})_0$$

Proof. This is an awfully large formula, but here is the idea for y. At a point $\lambda \in \mathfrak{h}^\star/W$, the terms in the Dunkl operator formula not inside W^λ are actually regular functions, so are already included in the completion algebra by means of the x-terms.

So the formula for y says something like "the Dunkl operator on the whole thing is the Dunkl operator for W^λ, plus terms that are regular functions on the quotient."

□

Definition. Finally, there is a duality functor

$$\dagger : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_c(W, \mathfrak{h}^\star)^{op}$$

Which sends a module to the restricted dual space under grading of generalised eigenvalues by h.

Now we can define induction and restriction functors.

Definition. Let

$$ind_\lambda : \mathcal{O}_{c'}(W_\lambda, \mathfrak{h}/\mathfrak{h}^{W^\lambda})_0 \to \mathcal{O}_c(W, \mathfrak{h})_0 N \to J_0 \circ \iota_\lambda \circ \psi_\lambda^{-1}(N)$$
$$res_\lambda : \mathcal{O}_c(W, \mathfrak{h})_0 \to \mathcal{O}_{c'}(W_\lambda, \mathfrak{h}/\mathfrak{h}^{W^\lambda})_0$$
$$M \to \psi_\lambda \circ J_\lambda \circ \iota_0(N)$$
$$Res_{\lambda^\star} := \dagger \circ res_\lambda \circ \dagger$$
$$Ind_{\lambda^\star} := \dagger \circ ind_\lambda \circ \dagger$$

where ι_λ denotes the inclusion of $\mathcal{O}_c(W, \mathfrak{h})_\lambda$ into $\mathcal{O}_c(W, \mathfrak{h})$.

Where $b = \lambda^\star$.

Claim. Ind_b is right adjoint to Res_b. ind_λ is left adjoint to res_λ. All these functors are exact.

Claim. If c is generic, res_λ descends to the usual restriction functor between $Rep(W)$ and $Rep(W^\lambda)$.

We can more globally define a functor

$$Res : \mathcal{O}_c(W, \mathfrak{h})_0 \to \mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'}) \boxtimes Loc(\mathfrak{h}_{reg}^{W'})$$

By upgrading b to a variable. The key thing we need to upgrade is the formula for the map θ. Rather than completing at the regular point (i.e. point where the formula for θ is nonsingular) b, we should now complete at the set of points fixed by a given stabiliser group, $\mathfrak{h}^{W'}$.

Given this choice of points for b, the targets are all isomorphic, but we should keep track of how they are so. This is a problem of parallel transport, and we should keep track of a local system in $Loc(\mathfrak{h}_{reg}^{W'})$.

A full construction of this partial KZ functor can be found in Seth Shelley-Abrahamson's thesis.

Remark. If W' is the trivial group, this is called the **Knizhnik-Zamolodchikov-functor**.

Remark. The aformentioned isomorphism with $Rep(W)$, plus an observation that I will not justify that these functors vary in a 'flat' way, implies that if c is generic, then

$$Res(M_c(\tau)) = \oplus_i M_c(W', \mathfrak{h}/\mathfrak{h}^{W'}, \tau_i) \otimes \mathcal{L}_{\tau,\tau_i}$$

Where $\mathcal{L}_{\tau,\tau_i}$ is a local system of rank n_{τ,τ_i}.

Claim. The local system is given by the trivial bundle with connection

$$\nabla = d - \sum_{s \in S, s \notin W'} \frac{c_s d\alpha_s}{\alpha_s}(1-s)$$

Example. Let $W = S_n$, acting on $\mathbb{C}^n$ via the permutation representation. Let $W' = S_{n_1} \times \cdots \times S_{n_k}$.

Then the KZ connection can be written

$$d - c \sum_{p \neq q, \sum_k^{p-1} n_k < i \leq \sum_k^p n_k, \sum_k^{q-1} n_k < j \leq sum_k^q n_k} \frac{dz_p - dz_q}{z_p - z_q}(1 - s_{ij})$$

A flat section of this equation satisfies

$$\partial_p F = c \sum_{p \neq q, \sum_k^{p-1} n_k < i \leq \sum_k^p n_k, \sum_k^{q-1} n_k < j \leq sum_k^q n_k} \frac{dz_p - dz_q}{z_p - z_q}(1 - s_{ij})$$

Conjugating $F \to G \prod_{p<q} (z_p - z_q)^{cn_p n_q}$, we get the equation

$$\partial_p G = -c \sum_{p \neq q, \sum_k^{p-1} n_k < i \leq \sum_k^p n_k, \sum_k^{q-1} n_k < j \leq sum_k^q n_k} \frac{s_{ij} G}{z_p - z_q}$$

which is the ordinary KZ equation, in the setting where all the subgroups are trivial.

Remark. In the case where $W' = 1$, this functor reduces to a functor $Res : \mathcal{O}_c(W, \mathfrak{h}])_0 \to Loc(\mathfrak{h}_{reg}/W) \simeq Rep(B_W)$, the latter isomorphism being the monodromy functor.

Then [GGOR] prove that the KZ functor factors through $Rep\mathcal{H}_q(W)$, for some special value of q.

The idea of the proof is to explicitly compute the monodromy of the KZ connection for special c, and use flatness to get it for all c.