DAHAAHAHA!

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↳ Hecke algs H_R (root system) : τ-deformed $\mathbb{Z}[W]$ (associated to R) $\quad (T_i - \tau_i)(T_i + \tau_i^{-1}) = 0$
$\quad\quad W \ni T_i$, $\quad \tau_i \leftarrow \tau$

↳ Affine Hecke algs $\mathcal{H}_R$: τ-deformed $\mathbb{Z}[W^{ae}]$ (extended affine W) $\quad W^{ae} = W \ltimes P^\vee = W^a \ltimes P^\vee/Q^\vee$, $\quad P^\vee/Q^\vee = \Omega$

Coxeter presentation

$\underbrace{T_0, T_1, \dots, T_r}_{\text{usual Hecke alg relations } (\mathbb{Z}[W^a])}, \ \pi \in \Omega$

$\pi T_i \pi^{-1} = T_j \quad$ if $\pi(\alpha_i) = \alpha_j$

Bernstein presentation

$\underbrace{T_1, \dots, T_r}_{\text{usual } H_R}, \ Y^{\lambda^\vee} \leftarrow \lambda^\vee \in P^\vee$

commutative $Y^{\lambda^\vee} Y^{\mu^\vee} = Y^{\lambda^\vee + \mu^\vee}$

$T_i Y^{\lambda^\vee} = Y^{\lambda^\vee} T_i \qquad \langle \lambda^\vee, \alpha_i \rangle = 0$

$T_i Y^{s_i(\lambda^\vee)} T_i = Y^{\lambda^\vee} \qquad \langle \lambda^\vee, \alpha_i \rangle = 1$

↳ Double affine Hecke alg : $\mathbb{H}_R$: (q,τ)-deformed $\mathbb{Z}[P \rtimes W \ltimes P^\vee]$

$T_1, \dots, T_r \qquad Y^{\lambda^\vee} \qquad X^\mu \leftarrow \mu \in P$

s.t. $\quad \langle T_1, \dots, T_r, Y^{\lambda^\vee} \rangle$ forms an AHA $\mathcal{H}_R$

$\langle T_1, \dots, T_r, X^\mu \rangle$ forms an AHA $\mathcal{H}_{R^\vee}$

$\pi X^\mu \pi^{-1} = X^{\pi(\mu)} \quad \leftarrow$ interaction between P and $P^\vee$

$X^{\lambda + r\delta} := q^r X^\lambda$

$\pi(\mu) \in P^a := P \oplus \mathbb{Z}\delta$

we'll capture the affine δ using new variable q

Why DAHA? One possible motivation is as follows.

Recall : polynomial rep of AHA $\mathcal{H}_{R^\vee} \curvearrowright \mathbb{C}[X^\mu] = \operatorname{ind}_H^{\mathcal{H}} \mathbb{C}$, $\quad \mathbb{C}[X^\mu] = \mathbb{C}[X]$

$T_i \longmapsto \tau_i s_i + (\tau_i - \tau_i^{-1}) \dfrac{\mathrm{id} - s_i}{1 - X^{\alpha_i}}$

$X^\mu \longmapsto X^\mu$

follows from relations in AHA

$\exists$ a basic rep of AHA $\mathcal{H}_R \curvearrowright \mathbb{C}[X] \otimes \mathbb{C}[q^{\pm}]$

$w \cdot X^\mu = X^{w(\mu)}$

$$\omega \cdot X^{\mu} = X^{\omega(\mu)}$$
$$\uparrow_{\in W^a}$$

same as T_i action in poly. rep.

$$T_i \longmapsto \tau_i s_i + (\tau_i - \tau_i^{-1}) \frac{\mathrm{id} - s_i}{1 - X^{\alpha_i}}$$

$$\pi \longmapsto \pi$$

$\Rightarrow$ glue to give the poly. rep. of DAHA $\mathcal{H}_R \curvearrowright \mathbb{C}[X]$

T_i = Demazure–Lusztig operators

"hard" $Y^{\lambda^\vee}$ = q-Dunkl operators

Importantly, eigenfunctions for W-invariant polys of Y's are Macdonald polynomials (in X).

"easy" X^{μ} = multiplication

$\exists$ an involution of DAHA (part of a whole $\widetilde{SL(2,\mathbb{Z})} \subset \mathrm{Aut}(\mathcal{H}_R)$)

$$\varepsilon : \quad X \leftrightarrow Y, \quad T_i \leftrightarrow T_i^{-1}, \quad \begin{matrix} \tau \leftrightarrow \tau^{-1} \\ q \leftrightarrow q^{-1} \end{matrix}$$

$\Rightarrow$ once we want to study $Y^{\lambda^\vee}$, it's natural to also include X^{μ}.

W-invariant

Often it's nice to symmetrize DAHA

Def: Let $\tilde{e} := \sum_{w \in W} \tau_w T_w$ be a "symmetrizer".

$= \tau_{i_1} \cdots \tau_{i_k}$ where $w = s_{i_1} \cdots s_{i_k}$

Compute: $T_i \tilde{e} = \tau_i \tilde{e}$

$$\Rightarrow \tilde{e}^2 = \Big(\underbrace{\sum_{w \in W} \tau_w^2}_{\neq}\Big) \tilde{e} \quad \Rightarrow \quad e := \tilde{e} \Big/ \sum_{w \in W} \tau_w \quad \text{is idempotent.}$$

The spherical DAHA is the sub-algebra (not unital)

new unit is e

$$S\mathcal{H}_R = e \mathcal{H}_R e \subset \mathcal{H}_R$$

If $\mathcal{H}_R \curvearrowright M$, then $S\mathcal{H}_R \curvearrowright eM$

e.g. Macdonald operators $\in S\mathcal{H}_R \curvearrowright e\,\mathbb{C}[X] = \mathbb{C}[X]^W$

home of symmetric polynomials.

Thm: (PBW decomposition)

$$\{ X^\lambda \pi T_w \mid \lambda \in P,\ \pi \in \Omega,\ w \in W^a \}$$

is a basis of $\mathcal{H}_R$.

Pf: Spanning ← easy, just commute things until all terms are of the desired form.

Linear independence ← suffices to prove in a representation.

↑ we'll use $\mathbb{C}[X]$

↑ conclude is faithful.

Key observation: Demazure–Lusztig operators have the form

$$T_i = (\cdots)\, s_i + (\cdots)\, id$$

$\neq 0$ rational functions of X's.

and so

$$T_w = \sum_{w' \le w} c_w^{w'}(X)\, w' \qquad \text{with} \quad c_w^w \neq 0.$$

$\Rightarrow$ Any relation $\sum_{\substack{w \in W^a \\ \pi \in \Omega}} g_{\pi,w}(X)\, \pi T_w = 0$ can be rewritten as

$$\sum_{\substack{w \in W^a \\ \pi \in \Omega \\ w' \le w}} g_{\pi,w}(X)\, c_w^{w'}(X)\, (\pi w') = 0.$$

↑ know are independent, since $W^a \curvearrowright \mathbb{C}[X]$ is faithful.

$\Rightarrow$ For each $\pi w'$,

$$\sum_{w \ge w'} g_{\pi,w}(X)\, c_w^{w'}(X) = 0$$

Pick w_{top} maximal wrt. $\ge$ such that $g_{\pi,w_{top}}(X) \neq 0$.

Then for $w' = w_{top}$

$$g_{\pi,w_{top}}\, \underbrace{c_{w_{top}}^{w_{top}}(X)}_{\neq} = 0 \qquad \text{—✗—}$$

$\neq 0$

$\square$