20210331-131117-0500–My talk: Schur-Weyl Duality

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Contents

[2021-03-31 Wed 13:11]

tag Schur-Weyl duality, double commutant theorem, representation theory

Disclaimer: this talk does not talk about Schur-Weyl duality for the quantum affine nor the quantum toroidal setting.

Reference

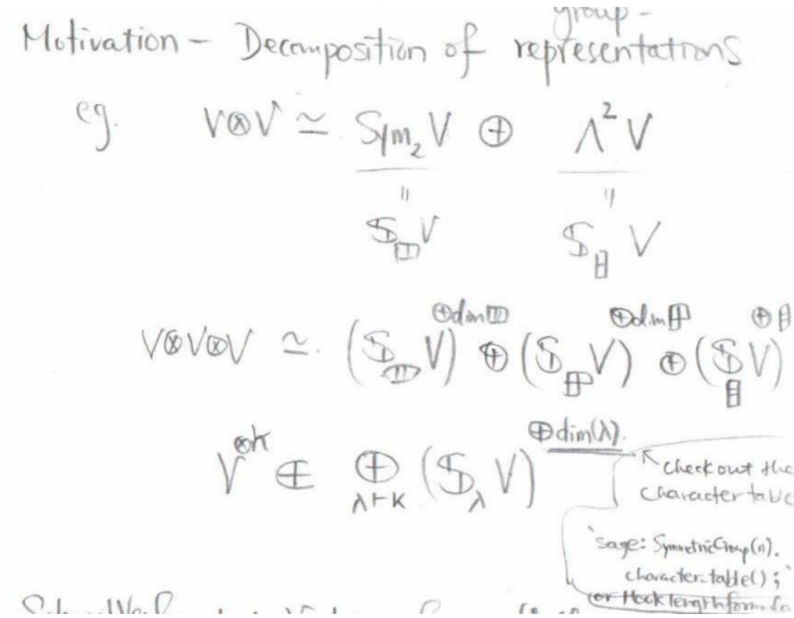
- [A] Drinfeld Functor and Finite-Dimensional Representations of Yangian-[Tomoyuki Arakawa]
- [C and S] Quantum affine algebras and affine Hecke algebras-[Vyjayanthi Chari and Andrew Pressley]-[arXiv:q-alg–9501003]
- [MNO] Yangians and classical Lie algebras-[Molev and Nazarov and Olshanski]
- [S] Schur Weyl Duality for Quantum Groups-[Yi Sun]
- [GW] Symmetry, Representations, and Invariants-[R. Goodman and N. Wallach]

Talk

Classical

Motivation **ATTACH**

- Decomposition of finite group representations



- For $\dim(\lambda)$, check out the character tables for the symmetric groups, or use the hook length formula.

```
for n in [1 .. 4]:
  print(n);
  SymmetricGroup(n).character_table();
  print("");
```

```
1
[1]

2
[ 1 -1]
[ 1  1]
```

```
3
[ 1 -1  1]
[ 2  0 -1]
[ 1  1  1]

4
[ 1 -1  1  1 -1]
[ 3 -1 -1  0  1]
[ 2  0  2 -1  0]
[ 3  1 -1  0 -1]
[ 1  1  1  1  1]
```

Schur-Weyl duality (classical) **ATTACH**

- The above statement can be refined to be a duality statement between the symmetric groups S_k and the reductive algebraic groups GL(n;C) or SL(n;C).

  This is the classical Schur-Weyl duality

Schur-Weyl: Let V be a finite dim'l (or Hook length formula) vector space, $n \in \mathbb{N}$. Then as a $S_k \otimes GL(n,\mathbb{C})$ module,

$$\otimes^k(\mathbb{C}^n) \simeq \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} (S_\lambda V) \boxtimes L_\lambda$$

where L_λ is the highest wt module of $SL(n,\mathbb{C})$ w/ weight $= (\lambda_1-\lambda_2)\omega_1 + (\lambda_2-\lambda_3)\omega_2 + \dots + (\lambda_{n-1}-\lambda_n)\omega_n$ and $zI \in GL(n,\mathbb{C})$ acts as z^k.

(ref. Goodman & Wallach thm 5.5.22)
thm 9.1.2

- Note that when $n \geq k$, all irreps of S_k show up in the direct sum!

Sketch of proof **ATTACH**

- As the proofs of the quantum cases all seem to based on this fact, I figure it would be nice to know a sketch of its proof.
- The idea is to use a general duality theorem, that requires the two algebras to be double commutant to each other.

  Double Commutant + Duality
- First, we have the classical Schur's double commutant theorem:

<u>Schur's Theorem (Double Commutant)</u> [Goodman+Wallach thm 4.2.10]

By the two actions, we have $S_K \hookrightarrow \mathrm{End}(\otimes^k \mathbb{C}^n) \hookleftarrow GL(n,\mathbb{C})$

For any subalg $A \subseteq \mathrm{End}(\otimes^k \mathbb{C}^n)$, denote its commutant be

$$A' := \{ x \in \mathrm{End}(\cdot) \mid ax = xa \quad \forall a \in A \}$$

Then $(S_K)' = GL(n,\mathbb{C})$ and $S_K = (GL(n,\mathbb{C}))'$.

- This allows us to use the following general duality theorem.

<u>Thm (General Duality)</u> [G+W, thm 4.2.1]

Let $G \subseteq GL(n,\mathbb{C})$ be a reductive linear algebraic group.

Let (ρ, L) be a locally regular rep of G w/ countable diml

Let R be a subalg of $\mathrm{End}(L)$ s.t. R acts irreducibly on L. Suppose $\rho(g) r \rho(g)^{-1} \in R$ $\forall g \in G\ \forall r \in R$, so $G \curvearrowright R$, and suppose that this action is also loc regular.

Then 1) $L \simeq \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} \underset{\underset{\mathrm{Hom}_G(F^\lambda, L)}{\shortparallel}}{E^\lambda} \otimes F^\lambda$ as an $R^G \otimes G$ module

2) E^λ is irrep of R^G, and $E^\lambda \leftrightarrow F^\lambda$ is a bijection on irreps of both that occur in L.

- Indeed,

p3/4

By these theorems, w/

$$L = \otimes^k(\mathbb{C}^n)$$

$$G = GL(n, \mathbb{C})$$

$$R = \mathrm{End}(L).$$

We have $R^G = \mathbb{C}[S_k]$, so

$$\otimes^k(\mathbb{C}^n) \simeq \bigoplus_{\lambda \in \widehat{(\otimes^k \mathbb{C}^n \circlearrowleft GL(n,\mathbb{C}))}} E^\lambda \boxtimes F^\lambda$$

as an $S_k \otimes GL(n,\mathbb{C})$-module.

Further analysis on weights and characters tells us that

$$\otimes^k(\mathbb{C}^n) \simeq \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} (\mathbb{S}_\lambda V) \boxtimes L_\lambda .$$

#

- Aside

  1. If you see an essentially different proof, please let me know.
  2. Before preparing this talk, I hoped the quantum cases can be proved by some very (more) general duality theorem. But all accounts I have do not mention such method.

     The general duality theorem I have here does require one side to be a reductive group, and the vector space to be a locally regular representation.

Remarks

- We have a coincidence - S_k is the Weyl group of type A.

This is merely a coincidence, as in type B, C, D, we don't have W on the other side. Instead, one should look for Brauer algebras and partition algebras [Halverson - 1605.06543].

- To pass to the quantum case, we first replace G = SL(n;C) by sl_n and then by $U(sl_n)$ – This does not change the representation theory.

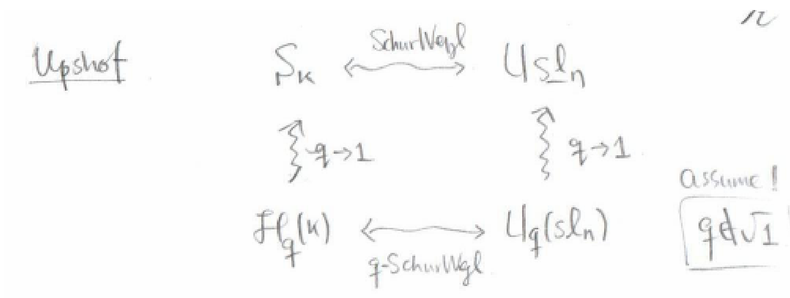
  Then we deform and get the quantum group $U_q(sl_n)$, for q not a root of unity.

Quantum (finite type)

In this section, we follow Yi Sun's note on Schur duality.

$U_q(sl_n)$ and Hecke algebra **ATTACH**

- Upshot



  Schur-Weyl duality extends to $U_q(sl_n)$!

- Compare the $Rep(U(sl_n))$ and $Rep(U_q(sl_n))$

  1. They are both monoidal, with isomorphic Grothendieck rings.
  2. However, they differ as monoidal categories (c.f. 6j symbols or F-symbols).
  3. As braided categories, they differ even more.

Recall that $\left(\begin{array}{c} V\otimes W \to W\otimes V \\ v\otimes w \mapsto w\otimes v \end{array} \right)$ is a morphism in $\mathrm{Rep}(U\mathfrak{sl}_n)$, but NOT IN $\mathrm{Rep}(U_q\mathfrak{sl}_n)$.

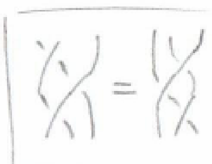
This boils down to the fact that $U\mathfrak{sl}_n$ is cocommutative, whereas $U_q\mathfrak{sl}_n$ isn't by design.

Nevertheless, $V\otimes W \simeq W\otimes V$ in both case, and in the $q\neq 1$ case it's realized by nontrivial R-matrices : $V\otimes W \xrightarrow[\sim]{\hat{R}_q} W\otimes V$

p7/2

Such that $\hat{R}^2 \neq 1$

$$\hat{R}^{23}\hat{R}^{12}\hat{R}^{23} = \hat{R}^{12}\hat{R}^{23}\hat{R}^{12} \quad (YB)$$



$$\hat{R}\big|_{\mathbb{C}^n\otimes\mathbb{C}^n} = \boxed{q\sum_i E_{ii}\otimes E_{ii} + \sum_{i\neq j} E_{ii}\otimes E_{jj} + (q-q^{-1})\sum_{i>j} E_{ij}\otimes E_{ji}}$$

4. The fact that $R^2 \neq 1$ stops S_k from acting on $\otimes^k(C^n)$.. instead, we have a Hecke algebra action:

$$\mathcal{H}_q(k) := \left\langle T_1, T_2, \dots, T_{k-1} \;\middle|\; \begin{array}{l} (T_i - q^{-1})(T_i + q) = 0 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\ [T_i, T_j] = 0 \ \text{ if } |i-j| \neq 1 \end{array} \right\rangle$$

by $T_i \mapsto \hat{R}^{i,i+1}\big|_{(\otimes^k \mathbb{C}^n)}$

Double commutant for $\mathbf{U_q(sl_n)}$ and $\mathbf{H_q(m)}$ **ATTACH**

(Double commutant)
Theorem 2.6(a) If $q \notin \sqrt{1}$, and if $n > m$, we have that the image of $U_q(sl_n)$ and the image of $\mathcal{H}_q(m)$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes m})$ are commutants of each other, where V denotes the std rep of $U_q(sl_n)$.

Proof **ATTACH**

- notation

Denote. $A_q = \text{image}(U_q sl_n) \subseteq \mathrm{End}(V^{\otimes m})$.
$B_q = \text{image}(H_q(m)) \subseteq \mathrm{End}(V^{\otimes m})$.

We will show that $B_q = A_q'$
and ~~that~~ $A_q = A_q''$.

- 1. $B_q = A_q$'

1.1) $\underline{B_q \subseteq A_q'}$: Immediate by def of the action.. one is from left & one is from right.

1.2) $\underline{B_q = A_q'}$: We will count dimensions. From the classical case,

31_130335.jpeg

we have $B_{q=1} = (A_{q=1})'$.

It remains to show that $\dim A_q = \dim A_1$

and $\dim B_q = \dim B_1$

– $\dim(A_q) = \dim(A_1)$

- $q \notin \sqrt{1} \Longrightarrow U_q(\mathfrak{sl}_n)$ semisimple "for finite dim'l modules" ref!? [Kassel-Quantum Group VII.7.2. (for $\mathfrak{sl}_2$)] [see also Kassel QG §5.17]

  so $\underset{U_q\text{mod}}{V^{\otimes m}} \simeq \bigoplus_a \underset{(\text{irrep})}{\underline{M_a}}$

  and $A_q \simeq \bigoplus_a \mathbb{C}_a$ by Schur's lemma.

- But the Grothendieck groups of $\mathrm{Rep}(U_q\mathfrak{sl}_n)$ and $\mathrm{Rep}(U_1\mathfrak{sl}_n)$ are the same (ref?),

  so $\dim(A_q) = \dim(A_1)$. or [Curtis–On Lusztig's isom thm for Hecke algs.]

henry: Uq(sl2) is not semisimple take evaluation representations. . . etc

– $\dim(B_q) = \dim(B_1)$

2) $\dim B_q = \dim B_1$

[Jimbo – A q-analog of $U(gl(N+1))$, proof of proposition 3]

- $q \notin \sqrt{1} \Rightarrow \mathcal{H}_q(m) \underset{alg!}{\simeq} \mathbb{C}[S_m]$ (ref. ?).
- $\mathcal{H}_q(m)$ acts on $V^{\otimes m}$ faithfully (why?)

  and $\mathbb{C}[S_m]$ too (trivial), so

  $\dim B_q = \dim B_1$.

...mpletes 1).

- 2. $A_q = A_q''$

2). $A_q = A_q''$

Since $U_q(sl_n)$ is semisimple, $V^{\otimes m}$ is a completely reducible $U_q(sl_n)$ module.

By a general algebra theorem (ref. Goodman & Wallach Thm 4.1.13)

$A_q = A_q''$.

This completes the proof of thm 2.6 (a).

Duality for $U_q(sl_n)$ and $H_q(m)$ **ATTACH**

- q-Schur-Weyl Duality

Theorem 2.6(b) If $q \notin \sqrt{1}$, we have that

$$V^{\otimes m} = \bigoplus_{\substack{\lambda \vdash m \\ \ell(\lambda) \leq n}} S_\lambda^q \boxtimes L_\lambda$$

as a $\mathcal{H}_q(m) \otimes U_q(\mathfrak{sl}_n)$ module.

Proof **ATTACH**

pf As mentioned in the previous proof, finite dim'l modules of $U_q(\mathfrak{sl}_n)$ are semisimple. So,

$$V^{\otimes m} \underset{U_q\text{-mod}}{=} \bigoplus_{\lambda \in \widehat{U_q}} \underbrace{\mathrm{Hom}_{U_q}(L_\lambda, V^{\otimes m})}_{\mathbb{C}\text{-vect}} \otimes \underbrace{L_\lambda}_{U_q\text{ mod}}.$$

Obviously, this decomposition is as a $\mathcal{H}_q(m) \otimes U_q(\mathfrak{sl}_n)$-module as well.

It remains to prove that $\mathrm{Hom}_{U_q}(L_\lambda, V^{\otimes m})$ as an $\mathcal{H}_q(m)$-module, it is the q-deformation of the $\mathbb{C}[S_m]$ module S_λ. pf/4

This is claimed to be true in [Yi: San] .. but IMHO a complete proof should ~~~~ be done by writing down an explicit map mimicking the classical case. (omit) #

Corollary **ATTACH**

Corollary 2.7. For $n > m$, the functor $\mathrm{Rep}(\mathcal{H}_q(m)) \xrightarrow{FS_q} \mathrm{Rep}(U_q \mathfrak{sl}_n)$, $W \mapsto \mathrm{Hom}_{\mathcal{H}_q(m)}(W, V^{\otimes m})$

is an equivalence of cats between $\mathrm{Rep}(\mathcal{H}_q(m))$ and the full subcat of $\underbrace{\text{weight } m}$ reps of $U_q(\mathfrak{sl}_n)$.

meaning those showing up in $V^{\otimes m}$.

- Proof

Pf By S.S., we have $V^{\otimes m} = \bigoplus_{\substack{\lambda \vdash m \\ \ell(\lambda) \le n}} S^q_\lambda \otimes \mathrm{Hom}_{\mathcal{H}_q(m)}(S^q_\lambda, V^{\otimes m})$ as an $\mathcal{H}_q(m)$-module.

By 2.6(b), we have $\mathrm{Hom}_{\mathcal{H}_q(m)}(S^q_\lambda, V^{\otimes m}) \underset{(U_q\text{-mod})}{\simeq} L_\lambda$, and also that FS_q is ess. onto the "wt m" subcat.

Since both S^q_λ and L_λ are simple, FS_q is indeed fully faithful. #

Quantum (Yangian) **ATTACH**

- reference

ref: [Yi Sun — Schur Weyl duality for Quantum Groups §3]

Informal talknote. compact.

[Molev, Nazarov, Olshanski — Yangians and Classical Lie algs §1 + §2]

main ref

[Arakawa — Drinfeld functor & fin' dim'l reps of Yangian]

Yangian of type A **ATTACH**

- Yang Baxter equation (with a spectral parameter)

$$R_{12}(u_1-u_2)\, R_{13}(u_1-u_3)\, R_{23}(u_2-u_3)$$
$$\|$$
$$R_{23}(u_2-u_3)\, R_{13}(u_1-u_3)\, R_{12}(u_1-u_2).$$

for $R(u) \in End(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes \mathbb{C}(u)$.

This is solved in terms of the Yangians of gl_n — $Y(gl_n)$.

- $Y(gl_n)$

<u>Def</u> $(Y(gl_n))$ As an algebra, it is

[MNO, 1.1] [1.8] [S. 3.1] $\left\langle t_{ij}^{(k)} \;\middle|\; \begin{array}{l} k = 0, 1, 2, \ldots \\ 1 \le i, j \le n \end{array} \right\rangle$

$$R^{12}(u-v)\, t^1(u)\, t^2(v) = t^2(v)\, t^1(u)\, R^{12}(u-v)$$

where $R = 1 - \frac{P}{u}$

$t(u) = \sum_{ij} t_{ij}(u) \otimes E_{ij}$

$t_{ij}(u) = \delta_{ij} u^{-1} + \sum_{k \ge 1} t_{ij}^{(k)} u^{-k-1}$

<u>Thm</u> Let i, j, k pairwise distinct.

- <u>Theorem</u> $Y(gl_n)$ solves the YB equation with one spectral parameter.
- Some remarks

1. $Y(gl_n)$ has a Hopf algebra structure

$$\Delta(t_{ij}(u)) = \sum_{a=1}^{n} t_{ia}(u) \otimes t_{aj}(u)$$
$$S(t(u)) = t(u)^{-1}$$

[MNO 1.28]

2. $\mathrm{U}(\mathrm{gl_n})$ embeds into Y as a Hopf algebra

$$\begin{array}{ccc} U(\mathfrak{gl}_n) & \xhookrightarrow[\text{Hopf}]{} & Y(\mathfrak{gl}_n). \\ E_{ij} & \longmapsto & t_{ij}^{(1)} \end{array}$$

$\begin{bmatrix} \text{MND } 1.17 \\ \text{S. } \S 3 \end{bmatrix}$

3. For type A, there's an exceptional algebra morphism going the other side.

$$\begin{array}{ccc} U(\mathfrak{gl}_n) & \xleftarrow[\text{alg.}]{ev_a} & Y(\mathfrak{gl}_n). \\ \delta_{ij} + \dfrac{E_{ij}}{u-a} & \longleftarrow\!\shortmid & t_{ij}(u) \end{array} \quad (\forall a \in \mathbb{C})$$

[MNO 1.16]

This map seems to be exceptional for type A [MNO §0].

4. $\mathrm{Y}(\mathrm{gl_n})$ is a deformation for the current lie algebra $\mathrm{U}(\mathrm{gl_n}[x])$.

- $\mathrm{Y}(\mathrm{sl_n})$

Let $f = 1 + f_1 u^{-1} + f_2 u^{-2} + \dots \in \mathbb{C}[[u^{-1}]]$

Then μ_f. $Y(\mathfrak{gl}_n) \longrightarrow Y(\mathfrak{gl}_n)$

$t(u) \longmapsto f(u)t(u)$

is an automorphism.

· Def $Y(\mathfrak{sl}_n) := \{ x \in Y(\mathfrak{gl}_n) \mid \mu_f(x) = x \ \forall f \in \mathbb{C}[[u^{-1}]] \text{ w/ } f_0 = 1 \}$

[S. 3.2]
[MNO §2]

This is a Hopf subalg of $Y(\mathfrak{gl}_n)$.

- Remarks

1. We have compatible embeddings

$$\begin{tikzcd} U sl_n \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & Y(sl_n) \arrow[d, hook] \\ U gl_n \arrow[r, hook] & Y(gl_n) \end{tikzcd}$$

2. Representations of weight m

Say a rep of $Y(sl_n)$ is of weight m if it is of wt m as a $U(sl_n)$-rep.

recall this means that it occurs in $(\mathbb{C}^n)^{\otimes m}$.

3. Structure of $\mathrm{Y(gl_n)}$

$$Y(sl_n) \simeq Y(gl_n) \big/ (q\det(t(u)) - 1)$$

$$Y(gl_n) \simeq \text{Center} \otimes Y(sl_n)$$

Schur-Weyl duality for $\mathbf{Y(sl_n)}$

NONE. See the final section.

Beyond (conclusion) **ATTACH**

- The stories that follow

Following this line, the duality should be generalized from $\mathfrak{g} = \mathfrak{sl}_n$

$$
\begin{tikzcd}
 & \mathfrak{g} \arrow[dl] \arrow[dr] & \\
U_q\mathfrak{g} \arrow[dr] & & Y(\mathfrak{g}) \arrow[dl] \\
 & U_q(\hat{\mathfrak{g}}) \arrow[d] & \\
 & \mathrm{toroidal}(\mathfrak{g}) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & S_k \arrow[dl] \arrow[dr] & \\
\mathcal{H}_k \arrow[dr] & & \text{degen. } \hat{\mathcal{H}}_k \arrow[dl] \\
 & \hat{\mathcal{H}}_k \arrow[d] & \\
 & \mathcal{H}\mathcal{H}_k &
\end{tikzcd}
$$

DAHA.

- Reference

  - For $Y(\mathfrak{g})$, I haven't seen a complete account.
  - For $U_q(\hat{\mathfrak{g}})$, [Chari + Pressley – Quantum affine algs and affine Hecke algs] could be a complete account. According to [A], its technique can be used to produce a proof for the Yangian case.
  - For toroidal($\mathfrak{g}$), the standard account is [Varagnolo, Vasserot – Schur duality in the toroidal setting] ← I haven't checked its details

- In the rest of this talk, we will give the Drinfeld functor that embeds Rep(degenerated AHA) into Rep($\mathrm{Y}(\mathrm{gl}_\mathrm{n})$).

1. (Degenerated AHA)

   11/3

   (Degen. affine Hecke algebra)
   type A_{k-1}.

   $$\Lambda_k \underset{(\text{vect})}{\simeq} \mathbb{C}[S_k] \otimes \underbrace{\mathrm{Sym}(\mathfrak{h}_{(sl_k)})}_{\simeq\, \mathrm{Poly}(\mathfrak{h}^*_{sl_k})}$$

   $$\underset{\text{alg}}{\simeq} \left\langle \begin{matrix} s_1,\dots,s_{k-1} \\ x_1,\dots,x_k \end{matrix} \middle/ \begin{matrix} s_i^2 = 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ [x_i, x_j] = 0 \\ s_i x_i - x_{i+1} s_i = 1 \\ [s_i, s_j] = [s_i, x_j] = 0 \text{ if } |i-j| \neq 1 \end{matrix} \right\rangle$$

   For a review of its representation, see [A. section 1.4.]
2. (Drinfeld functor)

   Drinfeld Functor [A. §2]

   · For a left Λ_k module M, we will give a $Y(gl_n)$-module.

   Define $$D_k(M) \underset{\text{vect.}}{:=} M \otimes (\mathbb{C}^n)^{\otimes \ell} \Big/ \sum_{i=1}^{k-1} \mathrm{Im}(s_i + 1)$$

   where $\mathrm{Im}(s_i+1) = (s_i+1)\left(M \otimes (\mathbb{C}^n)^{\otimes \ell}\right)$
   s_i acts on M by $S_k \hookrightarrow \Lambda_k$, and on $(\mathbb{C}^n)^{\otimes \ell}$ by permutation.
3. $\mathrm{Y}(\mathrm{gl}_\mathrm{n})$ action

$Y(\mathfrak{gl}_n)$ acts on $D_k(M)$ by [A. (2.2.1)]

$$t_{ab}^{(d)} \mapsto \sum_{i=1}^{k} (y_i)^d \otimes \tau_i(E_{ab})$$

$$\boxed{(y_i) := \Big(x_i - \sum_{j<i} S_{ij}\Big)}$$

"only acts on the ith of $\mathbb{C}^n$ in $(\mathbb{C}^n)^{\otimes k}$"

4. "State of the proof/theorem"

Theorem [A P.8]

For $n > k$, $\mathrm{Rep}(\Lambda_k) \hookrightarrow \mathrm{Rep}(Y(\mathfrak{sl}_n))$

Proof See "[D2]", which is in Russian and of 2-page long. Actually, there's no proof in that paper, but one can deduce a proof from "[CP2]", which works for an even more general case.

$$\boxed{\mathrm{Rep}(AHA_k) \hookrightarrow \mathrm{Rep}(U_q(\widehat{\mathfrak{sl}_n}))}$$