DAHA & Affine Springer Fibers

$T^*_{\mathfrak{b}} G/B \simeq \mathfrak{n} \subset \mathfrak{b}$

Recall: [Chriss-Ginzburg]

G reductive $\quad T \subset B \subset G$
$\quad\quad\quad\quad\quad$ Torus Borel

$$\begin{matrix} T^*G/B \\ \downarrow \\ \mathcal{N} \end{matrix} \quad \text{- Springer fibration}$$

$$Z = T^*G/B \times_{\mathcal{N}} T^*G/B =: \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$$

Thm:

- $K^{G \times \mathbb{C}^*}(Z) \simeq \mathcal{H}_q^{\mathrm{aff}}$ under convolution $\star : K(Z) \times K(Z) \to K(Z)$

$$Z^{\text{center}}(\mathcal{H}^{\mathrm{aff}}) \simeq R(G)_{\mathbb{C}[q^{\pm 1}]}$$

- Given $\bar{a} = (a, q_0) \in G \times \mathbb{C}^*$

$$i_* H_\bullet \left(\text{fiber of} \begin{pmatrix} \tilde{\mathcal{N}}^{\bar{a}} \\ \downarrow \\ \mathcal{N}^a \end{pmatrix} \right)_\chi \leftarrow \chi \text{ isotypic}$$

$\chi \in \mathrm{Rep}(\pi_1(S_\lambda)) \quad \mathcal{N}^a = \bigsqcup_\lambda S_\lambda$.

exhaust all simple modules of $\mathcal{H}^{\mathrm{aff}}$ "Centralizer (a & x)".
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \hat{S}_x$

i: fiber $\hookrightarrow \pi^{-1}(W)$ W = slice through $\mathbb{O} = G x$

Goal: do this for

$\mathcal{H}$	$\mathcal{H}$	$\mathcal{H}$
DAHA	trigonometric	rational

following [Varagnolo, Varagnolo-Vasserot, Oblomkov-Yun, ...]

Basic idea: Replace G w/ LG & the proof carries through w/ lots of work for $\mathcal{H}$.

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§ Affine Springer Fibers / Affine flag variety

[Excellent ref. Z. Yun PCMI notes]

G/B parametrizes Borels $B \subset G$, or flags & $B = \mathrm{Stab}(\text{flag})$

G/P ——— Parabolics $P \subset G$, partial flags $P = \mathrm{Stab}(\text{flag})$.

Def loop group $LG = G((t))$

is an ind-scheme: $LG = \bigcup t^{-n} L^+G$

$L^+G = G[[t]]$ is

an infinite type scheme e.g. $\left(\sum_{n \geq 0} a_n^{ij} t^n \right)$, a_n^{ij} are coordinates

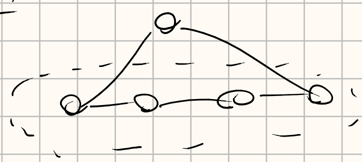
Def: $\mathbb{I} = ev_0^{-1}(B) \subset L^+G$ and its conjugates are called <u>Iwahori subgroups</u> (positive half + torus)

Then <u>Parahoric</u> subgroups $\mathbb{P}$

- contain an $\mathbb{I}$ w/ finite codimension
- have conjugacy classes labelled by $J \subsetneq I$ $\quad I$ = vertices of Dynkin diagram
- Fit into

$$0 \to \underset{\text{unipotent}}{\mathcal{U}_{\mathbb{P}}} \to \mathbb{P} \to \underset{\text{Levi}}{L_{\mathbb{P}}} \to 0$$

<u>Eg</u>:

1)  $:= J$

$\mathbb{P} = L^+G$

2) $J = \emptyset$

$\mathbb{P} = \mathbb{I}$

- J = Dynkin diagram of $L_{\mathbb{P}}$.

Def: $Fl_{\mathbb{P}} := LG/\mathbb{P}$ is the affine partial flag variety

Eg: $Fl_G = LG/\mathbb{I}$ full flag

$Gr_G = G((t))/G[[t]] = Fl_{L^+G}$ is the Affine grassmanian

<u>Facts</u>: • $\mathbb{P} \subset \mathbb{Q} \Rightarrow F \hookrightarrow Fl_{\mathbb{P}}$

$$\begin{array}{c} Fl_{\mathbb{P}} \\ \downarrow \\ Fl_{\mathbb{Q}} \end{array}$$

F = Partial flags in $L_{\mathbb{Q}}$ w/ Parabolic $im(\mathbb{P}) \subset L_{\mathbb{Q}}$.

- $Fl_{\mathbb{P}}$ is an ind-Scheme. Eg:

$Fl \supset \tilde{I} = I \rtimes \mathbb{C}^*_{loop} \times \mathbb{C}^*_{central}$ w/ finite type fixed loci,

and $Fl = \bigsqcup_{w \in W^a} Fl_w$.

• In (some?) classical case, $Fl_{\mathbb{P}}$ parametrizes lattices:

Eg: SL_n. Let $\Lambda = \{ \mathcal{O}^n \simeq \Lambda \hookrightarrow K^n \mid (t^{m}\mathcal{O})^n \subset \Lambda \subset (t^{-m}\mathcal{O})^n \}$

so $\Lambda \simeq G(K)/G(\mathcal{O}) = Gr_G$.

A J-periodic chain for $\emptyset \neq J \subset \mathrm{Vertices}(\tilde{A}_{n-1}) \simeq \mathbb{Z}/n\mathbb{Z}$ is

• $\cdots \subset \Lambda_{j_i} \subset \cdots \subset \Lambda_{j_k} \subset \cdots$ $\forall j_i \bmod n \in J$

• $\Lambda_{j_i+n} = t\Lambda_{j_i}$

• $[\Lambda_j : \mathcal{O}^n] = \dim(\Lambda_j/\Lambda_j\cap\mathcal{O}^n) - \dim(\mathcal{O}^n/\Lambda_j\cap\mathcal{O}^n) = j$

Then $\mathbb{P}_J \subset LG$ is the stabilizer of the chain $\Rightarrow$

$Fl_{\mathbb{P}}$ is moduli of chains.

Def: for $\gamma \in \mathfrak{g}((t))$ regular semisimple [i.e. $\pi(\gamma) \subset \mathfrak{g}^{G,rs}((t)) = \mathfrak{t}^{W,rs}((t))$]

the Affine Springer fiber is $(\mathcal{B}_\gamma^{nred})^{red} = \mathcal{B}_\gamma$

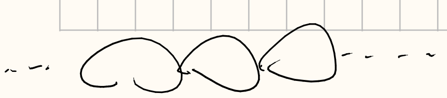
$\mathcal{B}_\gamma^{nred} = \{[g] \in Fl_{\mathbb{P}} \mid g\gamma g^{-1} \in \mathrm{Lie}\,\mathbb{P}\}$

Rmk: • Should be $\{\Lambda \in \Lambda_J \mid \gamma\Lambda_j \subset \Lambda_j\}$ (I'm only sure of this for Gr_G)

• Also an ind-scheme, but frequently (always?) of bounded dimension

Eg: $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \in \mathfrak{sl}_2((t))$

$\mathcal{B}_\gamma = \bigcup_{n\in\mathbb{Z}} \mathbb{P}^1_n$ $\quad \mathbb{P}^1_n = \{\Lambda \mid t^n\mathcal{O} \oplus t^{-n+1}\mathcal{O} \subset \Lambda \subset t^{n-1}\mathcal{O} \oplus t^{-n}\mathcal{O}\}$



§ Convolution algebras Weill do $\mathcal{H} = \mathcal{F}l_{Full}$

for simplicity

Problem: The scheme

$Z = T^*\mathcal{F}l \times_{\tilde{n}} T^*\mathcal{F}l$ is too infinite dimensional
and singular to define a convolution product
on $K^{G \times \cdots}(Z)$ directly

two solutions:

1) [V: induced & simple ...] Only define product on
fixed point sets $\coprod Z^A$.

2) [VV: DAHA & Affine Flag ...]
Use $K^{\hat{I} \times \mathbb{C}^*}(T^*\mathcal{F}l)$ instead.

They are equivalent [VV]. There's probably a diagram and the fact that $K^G(G \times_H M) \simeq K^H(M)$ that proves this
I couldn't quite work it out.

Basically the product is

$$T^*LG/I \times T^*LG/I \xleftarrow{\pi} \pi^{-1}(T^*)LG \times T^*LG/I \xrightarrow{m} T^*LG/I$$

where π is projection by $/I$ quotient & m is mult.

This takes lots of work!

§ DAHA Rep & Simple modules

We want a geometric construction of $\mathcal{H}$ as

$K^{\ddot{\mathbb{I}}}(T^*\mathcal{F}l)$ where, $T^*\mathcal{F}l = \bigsqcup_{w \in \tilde{W}} T^*\mathcal{F}l_w$ ← Bruhat stratification.

$\ddot{\mathbb{I}} = I \times \mathbb{C}^*_{loop} \times \mathbb{C}^*_{central} \times \mathbb{C}^*_{fiber}$

$T^*\mathcal{F}l = \{(x, \mathfrak{b}) \in \mathfrak{n} \times \mathcal{F}l \mid x \in \mathfrak{b}_{nil}\}$

First: given $\lambda \in R(T)$ we get $\mathcal{L}_\lambda$ by a scalar

$\mathcal{L}_\lambda \downarrow T^*\mathcal{F}l$

argument to usual: i.e. $\mathbb{C} \downarrow G \rightsquigarrow \mathcal{L}_\lambda \downarrow G/B$ with character $B \to B/_{[B,B]} = T \xrightarrow{\lambda} \mathrm{Aut}(\mathbb{C})$.

(really we do this over the Kashiwara flag manifold $\mathcal{X} \supset \mathcal{F}l$)

$\mathbb{C}_{q,t}[X]$ $\quad X_\lambda \mapsto \star[\mathcal{L}_\lambda]$ $\quad (= \star\, T^*\mathcal{F}l_w(\lambda)$ for $w = e$ $)$

as λ ranges over $R(T)$ gives the $\mathbb{C}_{q,t}[X] \subset \mathcal{H}$.

$\mathcal{H}_{R_Y}$ $\quad$ Given simple root α

$$T_{s_\alpha} \mapsto -[\mathcal{O}_{T^*\mathcal{F}l'_{s_\alpha}}(\mu)] \star [\mathcal{L}_\lambda] \text{ if } {}^{-1}$$

$\lambda + \mu = -\alpha$, $\langle \lambda, \check{\alpha} \rangle = \langle \mu, \check{\alpha} \rangle = -1$

where

$T^*\mathcal{F}l'_{s_\alpha}$ = sub-bundle of $T^*\mathcal{F}l_{s_\alpha}$ w/ fiber

$\mathfrak{n}(s_\alpha) = s_\alpha(\mathfrak{n}) \cap \mathfrak{n}$

$\mathbb{C}_{q,t}[Y]$ — act via base ring $K^{\ddot{I}}(pt) \simeq K^{\ddot{\mathbb{I}}}(pt)$.

Thm These assignments give an isomorphism

$$\Phi : \mathcal{H} \xrightarrow{\sim} K^{\Pi}(T^*\mathcal{Fl})$$

Pf idea: We do calculations in $T^*\mathcal{X}$ which has T-fixed points $p_{w,n}$

$$K^T(T^*\mathcal{X}^T) = \prod_w \left(\bigoplus_n R^T p_{w,n} \right) \quad \left(\begin{array}{c} \text{localization} \\ \text{key step} \end{array} \right)$$

Here $X_\lambda \mapsto \sum_{w \in \tilde{W}} \chi_{w\lambda}\, p_{w,w}$

$$T_{s_\alpha} \mapsto -1 - \sum_{\tilde{W}} \chi_{-w\alpha}\, p_{w,w} + p_{w, ws_\alpha}$$

$Y_{\lambda^\vee} \mapsto$ Base ring

Then, e.g. $p_{u,v} * p_{w,x} = \delta_{vw}\, p_{u,x}$ so X_λ commute.

Other relations are not so bad.

Thm: Just as in the affine case:

Given
- $(s, \tau, \xi) \in \hat{T} \times \mathbb{C}^*_{rot} \times \mathbb{C}^*_{filter}$ regular $\left(\tau^k \neq \xi^m \;\; k, m > 0 \right)$
- (χ, x) s.t. $\chi \in \mathrm{Irr}(\pi_1(S_{s\tau\xi}, x))$ occurs in the decomposition of a fiber $H_\bullet(\pi^{-1}(x), \mathbb{C})$

These exhaust simple modules and are isomorphic $\Leftrightarrow$ (s, τ, ξ, χ, x) are $\tilde{G}$-conjugate.

§ $\mathbb{H}$, $\mathbb{H}$, Springer fibers and Hitchin fibers

If the point x corresponds to a <u>Homogeneous</u> point in $\mathfrak{g}^G((t)) = \mathfrak{t}^W((t)) = \mathbb{C}[a_1^{d_1}, \dots, a_n^{d_n}]((t))$ of slope $\nu = n/m$

d_i = degrees

$$\mathbb{G}_\nu = \begin{array}{l} t \mapsto t^{1/m} \\ g \mapsto ng \\ \hat{g} \end{array} \quad \text{fixes } x.$$

[O, Y] In this case, $\pi^{-1}_{(\mathfrak{z},\tau,\xi)}(x)$ is a Springer fiber

Thm: • $\mathbb{H}_{\nu,\varepsilon}$ acts on $H_{G_\nu}(\mathcal{B}_x)$, and on $H(\mathcal{F}l)$

• If m is a regular number for W, (i.e. ν elliptic) then $\mathrm{im}\left(H(\mathcal{F}l) \to H(\mathcal{B}_x)\right) = H(\mathcal{B}_x)^{S \times B}$ irreducible quotient of the polynomial rep of $\mathbb{H}$ in $\mathcal{O}(\mathbb{H})$.

• A map $\mathcal{B}_x \longrightarrow \mathcal{HF}_a$ to the Hitchin fiber (In this case) produces a perverse filtration on $H(\mathcal{B}_x)^{S \times B}$ whose associated graded $\mathrm{Gr}^P H(\mathcal{B}_\nu)^{S \times B}$ is the finite-dimensional spherical module for $\mathbb{H}$.