3 Contour integrals and Cauchy's Theorem

3.1 Line integrals of complex functions

Our goal here will be to discuss integration of complex functions $f(z) = u + iv$, with particular regard to analytic functions. Of course, one way to think of integration is as antidifferentiation. But we also have the definite integral. For a function $f(x)$ of a real variable x, we have the integral $\int_a^b f(x)\,dx$. For vector fields $\mathbf{F} = (P, Q)$ in the plane we have the line integral $\int_C P\,dx + Q\,dy$, where C is an oriented curve. Let us begin by recalling the basics of line integrals in the plane:

1. The vector field $\mathbf{F} = (P, Q)$ is a gradient vector field ∇g, which we can write in terms of 1-forms as $P\,dx + Q\,dy = dg$, if and only if $\int_C P\,dx+Q\,dy$ only depends on the endpoints of C, equivalently if and only if $\int_C P\,dx+Q\,dy = 0$ for every closed curve C. If $P\,dx+Q\,dy = dg$, and C has endpoints z_0 and z_1, then we have the formula
$$\int_C P\,dx + Q\,dy = \int_C dg = g(z_1) - g(z_0).$$
2. If D is a plane region with oriented boundary $\partial D = C$, then
$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$
3. If D is a simply connected plane region, then $\mathbf{F} = (P, Q)$ is a gradient vector field ∇g if and only if $\mathbf{F}$ satisfies the mixed partials condition $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

(Recall that a region D is *simply connected* if every simple closed curve in D is the boundary of a region contained in D. Thus a disk $\{z \in \mathbb{C} : |z| < 1\}$ is simply connected, whereas a "ring" such as $\{z \in \mathbb{C} : 1 < |z| < 2\}$ is not.) Suppose that P and Q are complex-valued. Then all of the above still makes sense, and in particular Green's theorem is still true.

We will be interested in the following integrals. Let $dz = dx + idy$, a complex 1-form, and let $f(z) = u + iv$. Then we can define $\int_C f(z)\,dz$ for any reasonable closed oriented curve C. If C is a parametrized curve given by $\mathbf{r}(t)$, $a \le t \le b$, then we can view $\mathbf{r}'(t)$ as a complex-valued curve, and then

$$\int_C f(z)\,dz = \int_a^b f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt,$$

where the indicated multiplication is multiplication of complex numbers (and **not** the dot product). Another notation which is frequently used is the following. We denote a parametrized curve in the complex plane by $z(t)$, $a \le t \le b$, and its derivative by $z'(t)$. Then

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt.$$

For example, let C be the curve parametrized by $\mathbf{r}(t) = t + 2t^2 i$, $0 \le t \le 1$, and let $f(z) = z^2$. Then

$$\begin{aligned}\int_C z^2\,dz &= \int_0^1 (t+2t^2 i)^2(1+4ti)\,dt = \int_0^1 (t^2-4t^4+4t^3 i)(1+4ti)\,dt \\ &= \int_0^1 [(t^2-4t^4-16t^4)+i(4t^3+4t^3-16t^5)]\,dt \\ &= t^3/3 - 4t^5 + i(2t^4 - 8t^6/3)]_0^1 = -11/3 + (-2/3)i.\end{aligned}$$

For another example, let let C be the unit circle, which can be efficiently parametrized as $\mathbf{r}(t) = e^{it} = \cos t + i\sin t$, $0 \le t \le 2\pi$, and let $f(z) = \bar{z}$. Then

$$\mathbf{r}'(t) = -\sin t + i\cos t = i(\cos t + i\sin t) = ie^{it}.$$

Note that this is what we would get by the usual calculation of $\dfrac{d}{dt}e^{it}$. Then

$$\int_C \bar{z}\,dz = \int_0^{2\pi} \overline{e^{it}} \cdot ie^{it}\,dt = \int_0^{2\pi} e^{-it} \cdot ie^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

3.2 Cauchy's theorem

Suppose now that C is a simple closed curve which is the boundary ∂D of a region in $\mathbb{C}$. We want to apply Green's theorem to the integral $\int_C f(z)\,dz$. Working this out, since

$$f(z)\,dz = (u+iv)(dx+idy) = (u\,dx - v\,dy) + i(v\,dx + u\,dy),$$

we see that

$$\int_C f(z)\,dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA.$$

Thus, the integrand is always zero if and only if the following equations hold:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}; \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

Of course, these are just the Cauchy-Riemann equations! This gives:

Theorem (Cauchy's integral theorem): Let C be a simple closed curve which is the boundary ∂D of a region in $\mathbb{C}$. Let $f(z)$ be analytic in D. Then

$$\int_C f(z)\,dz = 0.$$

Actually, there is a stronger result:

Theorem (Cauchy's integral theorem): Let D be a simply connected region in $\mathbb{C}$ and let C be a closed curve contained in D. Let $f(z)$ be analytic in D. Then

$$\int_C f(z)\,dz = 0.$$

Example: let $D = \mathbb{C}$ and let $f(z)$ be the function $z^2 + z + 1$. Let C be the unit circle. Then as before we use the parametrization of the unit circle given by $\mathbf{r}(t) = e^{it}$, $0 \le t \le 2\pi$, and $\mathbf{r}'(t) = ie^{it}$. Thus

$$\int_C f(z)\,dz = \int_0^{2\pi} (e^{2it} + e^{it} + 1)ie^{it}\,dt = i\int_0^{2\pi} (e^{3it} + e^{2it} + e^{it})\,dt.$$

It is easy to check directly that this integral is 0, for example because terms such as $\int_0^{2\pi} \cos 3t\,dt$ (or the same integral with $\cos 3t$ replaced by $\sin 3t$ or $\cos 2t$, etc.) are all zero.

On the other hand, again with C the unit circle,

$$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} e^{-it} ie^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i \neq 0.$$

The difference is that $1/z$ is analytic in the region $\mathbb{C} - \{0\} = \{z \in \mathbb{C} : z \neq 0\}$, but this region is not simply connected. (Why not?)

Actually, the converse to Cauchy's theorem is also true: if $\int_C f(z)\,dz = 0$ for every closed curve in a region D (simply connected or not), then $f(z)$ is analytic in D. However, we will not discuss this.

3.3 Antiderivatives

We now look at the other method for showing that a line integral depends only on the endpoints. Let $f(z) = u + iv$ and suppose that $f(z)\,dz = dF$, where we write F in terms of its real and imaginary parts as $F = U + iV$. This says that

$$(u\,dx - v\,dy) + i(v\,dx + u\,dy) = \left(\frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy\right) + i\left(\frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy\right).$$

Equating terms, this says that

$$\begin{aligned} u &= \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \\ v &= -\frac{\partial U}{\partial y} = \frac{\partial V}{\partial x}. \end{aligned}$$

In particular, we see that F satisfies the Cauchy-Riemann equations, and its complex derivative is

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = f(z).$$

Thus we see:

Theorem: The 1-form $f(z)\,dz$ is of the form dF, or equivalently the vector field $(u + iv, -v + iu)$ is a gradient vector field $\nabla(U + iV)$, if and only if F is also analytic, and in this case $F(z)$ is an antiderivative for $f(z)$: $F'(z) = f(z)$. Conversely, if $F(z)$ is an antiderivative for $f(z)$, then $f(z)\,dz = dF$.

The theorem tells us a little more: if C has endpoints z_0 and z_1, and is oriented so that z_0 is the starting point and z_1 the endpoint, then we have the formula

$$\int_C f(z)\,dz = \int_C dF = F(z_1) - F(z_0).$$

For example, we have seen that, if C is the curve parametrized by $\mathbf{r}(t) = t+2t^2 i$, $0 \le t \le 1$ and $f(z) = z^2$, then $\int_C z^2\,dz = -11/3+(-2/3)i$. But $z^3/3$ is clearly an antiderivative for z^2, and C has starting point 0 and endpoint $1 + 2i$. Hence

$$\int_C z^2\,dz = (1 + 2i)^3/3 - 0 = (1 + 6i - 12 - 8i)/3 = (-11 - 2i)/3,$$

which agrees with the previous calculation.

If $f(z)$ is analytic in a **simply connected** region D, then the fact that $f(z)\,dz = P\,dx + Q\,dy$ satisfies $\partial Q/\partial x = \partial P/\partial y$ (here P and Q are complex valued) says that (P, Q) is a gradient vector field, or equivalently that $f(z)\,dz = dg$, in other words that $f(z)$ has an antiderivative.

From this point of view, we can see why $\displaystyle\int_C \frac{1}{z}\,dz = 2\pi i \neq 0$, where C is the unit circle. The antiderivative of $1/z$ is $\log z$, and so the expected answer (viewing the unit circle as starting at $1 = e^0$ and ending at $e^{2\pi i} = 1$ is $\log 1 - \log 1$. But log is not a single-valued function, and in fact as $z = e^{it}$ turns along the unit circle, the value of log changes by $2\pi i$. So the correct answer is really $\log 1 - \log 1$, viewed as $\log e^{2\pi i} - \log e^0 = 2\pi i - 0 = 2\pi i$. Of course, $1/z$ is analytic except at the origin, but $\{z \in \mathbb{C} : z \neq 0\}$ is not simply connected, and so $1/z$ need not have an antiderivative.

The real point, however, in the above example is something special about $\log z$, or $1/z$, but not the fact that $1/z$ fails to be defined at the origin. We could have looked at other negative powers of z, say z^n where n is a negative integer less than -1, or in fact any integer $\neq -1$. In this case, z^n has an antiderivative $z^{n+1}/(n+1)$, and so by the fundamental theorem for line integrals $\displaystyle\int_C z^n\,dz = 0$ for every closed curve C. To see this directly for the case $n = -2$ and the unit circle C,

$$\int_C z^{-2}\,dz = \int_0^{2\pi} e^{-2it} i e^{it}\,dt = i\int_0^{2\pi} e^{-it}\,dt = 0.$$

This calculation can be done somewhat more efficiently as follows. Let $\mathbf{r}(t) = e^{\alpha t}$, where α is a nonzero complex number. Then an antiderivative for the complex curve $\mathbf{r}(t)$ is checked to be

$$\mathbf{s}(t) = \int e^{\alpha t}\,dt = \frac{1}{\alpha}e^{\alpha t}.$$

Hence,

$$\int_a^b e^{\alpha t}\,dt = \frac{1}{\alpha}\left(e^{\alpha b} - e^{\alpha a}\right).$$

In general, we have seen that $\displaystyle\int_C z^n\,dz = 0$ for every integer $n \neq -1$, where C is a closed curve. To verify this for the case of the unit circle, we have

$$\begin{aligned}\int_C z^n\,dz &= \int_0^{2\pi} e^{nit} i e^{it}\,dt = i\int_0^{2\pi} e^{(n+1)it}\,dt \\ &= \frac{i}{n+1}\left(e^{2(n+1)\pi i} - e^0\right) = \frac{i}{n+1}(1-1) = 0.\end{aligned}$$

Finally, returning to $1/z$, a calculation shows that

$$\frac{1}{z}\,dz = \left(\frac{x\,dx}{x^2+y^2} + \frac{y\,dy}{x^2+y^2}\right) + i\left(\frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}\right).$$

The real part is the gradient of the function $\frac{1}{2}\ln(x^2+y^2)$. But the imaginary part corresponds to the vector field

$$\mathbf{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right),$$

which as we have seen many times this semester is a vector field $\mathbf{F}$ for which Green's theorem fails, because $\mathbf{F}$ is undefined at the origin. In the next section, we will see how to systematically use the fact that the integral of $1/z\,dz$ around a closed curve enclosing the origin to get a formula for the value of an analytic function in terms of an integral.

3.4 Cauchy's integral formula

Let C be a simple closed curve in $\mathbb{C}$. Then $C = \partial R$ for some region R. If z_0 is a point which does not lie on C, we say that C *encloses* z_0 if $z_0 \in R$, and that C *does not enclose* z_0 if $z_0 \notin R$. For example, if C is the unit circle, then C is the boundary of the unit disk $B = \{z : |z| < 1\}$. Thus C encloses a point z_0 if z_0 lies inside the unit disk ($|z_0| < 1$), and C does not enclose z_0 if z_0 lies outside the unit disk ($|z_0| > 1$).

Theorem (Cauchy's integral formula): Let D be a simply connected region in $\mathbb{C}$ and let C be a closed curve contained in D. Let $f(z)$ be analytic in D. Suppose that z_0 is a point enclosed by C. Then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz.$$

Before we discuss the proof, let us look at the special case where $f(z)$ is the constant function 1, C is the unit circle, and $z_0 = 0$. The theorem says in this case that

$$1 = f(0) = \frac{1}{2\pi i}\int_C \frac{1}{z}\,dz,$$

as we have seen. In fact, the theorem is true for a circle of any radius: if C_r is a circle of radius r centered at 0, then C_r can be parametrized by re^{it}, $0 \le t \le 2\pi$. Then

$$\int_{C_r} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i,$$

independent of r. The fact that $\int_{C_r} \frac{1}{z}\,dz$ is independent of r also follows from Green's theorem.

The general case is obtained from this special case as follows. Let $C = \partial R$, with $R \subseteq D$ since D is simply connected. We know that C encloses z_0, which says that $z_0 \in R$. Let C_r be a circle of radius r with center z_0. If r is small enough, C_r will be contained in R, as will the ball B_r of radius r with center z_0. Let R_r be the region obtained by deleting B_r from R. Then $\partial R_r = C - C_r$, where this is to be understood as saying that the boundary of R_r has two pieces: one is C with the usual orientation coming from the fact that C is the boundary of R, and the other is C_r with the **clockwise** orientation, which we record by putting a minus sign in front of C_r. Now z_0 does not lie in R_r, so we can apply Green's theorem to the function $f(z)/(z-z_0)$ which is analytic in D except at z_0 and hence in R_r:

$$\int_{\partial R_r} \frac{f(z)}{z-z_0}\,dz = 0.$$

But we have seen that $\partial R_r = C - C_r$, so this says that

$$\int_C \frac{f(z)}{z-z_0}\,dz - \int_{C_r} \frac{f(z)}{z-z_0}\,dz = 0,$$

or in other words that

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_r} \frac{f(z)}{z-z_0}\,dz.$$

Now suppose that r is small, so that $f(z)$ is approximately equal to $f(z_0)$ on C_r. Then the second integral $\int_{C_r} \frac{f(z)}{z-z_0}\,dz$ is approximately equal to

$$\int_{C_r} \frac{f(z_0)}{z-z_0}\,dz = f(z_0)\int_{C_r} \frac{1}{z-z_0}\,dz,$$

where C_r is a circle of radius r centered at z_0. Thus we can parametrize C_r by $z_0 + re^{it}$, $0 \le t \le 2\pi$, and

$$\int_{C_r} \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i,$$

as before. Thus

$$f(z_0)\int_{C_r} \frac{1}{z-z_0}\,dz = 2\pi i f(z_0),$$

and so $\displaystyle\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_r} \frac{f(z)}{z-z_0}\,dz$ is approximately equal to $2\pi i f(z_0)$. In fact, this becomes an equality in the limit as $r \to 0$. But $\displaystyle\int_C \frac{f(z)}{z-z_0}\,dz$ is independent of r, and so in fact

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0).$$

Dividing through by $2\pi i$ gives Cauchy's formula.

The main application of Cauchy's theorem is to think of the point z_0 as a **variable** point inside of the region R such that $C = \partial R$; note that the z in the formula is a dummy variable. Thus we could equally well write:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\,dw,$$

for all z enclosed by C. This description of the analytic function $f(z)$ by an integral depending only on its values on the boundary curve of R turns out to have many very surprising consequences. For example, it turns out that an analytic function actually has derivatives of all orders, not just first derivatives, which is very unlike the situation for functions of a reaal variable. In fact, every analytic function can be expressed as a power series.

3.5 Homework

1. Let $f(z) = x^2 + iy^2$. Evaluate $\displaystyle\int_C f(z)\,dz$, where (a) C is the straight line joining 1 to $2+i$; (b) C is the curve $(1+t)+t^2 i$, $0 \le t \le 1$. Are the results the same? Why or why not might you expect this?

2. Let $\alpha = c + di$ be a complex number. Verify directly that

$$\frac{d}{dt} e^{\alpha t} = \alpha e^{\alpha t}.$$

3. Let C be a circle centered at $4+i$ of radius 1. Without any calculation, explain why $\displaystyle\int_C \frac{1}{z}\,dz = 0$.

4. Let C be the curve defined parametrically as follows:

$$z(t) = t(1-t)e^t + [\cos(2\pi t^3)]i, \qquad 0 \le t \le 1.$$

Evaluate the integral $\displaystyle\int_C e^{z^2}\,dz$. Be sure to explain your reasoning!

5. Let D be a simply connected region in $\mathbb{C}$ and let C be a closed curve contained in D. Let $f(z)$ be analytic in D. Suppose that z_0 is a point which is **not** enclosed by C. What is $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-z_0}\,dz$?

6. Use Cauchy's formula to evaluate $\displaystyle\int_C \frac{e^z}{z+1}\,dz$, where C is a circle of radius 4 centered at the origin (and oriented counterclockwise).

7. Let D be a simply connected region in $\mathbb{C}$ and let C be a closed curve contained in D. Let $f(z)$ be analytic in D. Suppose that z_0 is a point enclosed by C.

   (a) By the usual formulas, show that

   $$\frac{d}{dz}\left(\frac{f(z)}{z-z_0}\right) = \frac{f'(z)}{z-z_0} - \frac{f(z)}{(z-z_0)^2}.$$

   (b) By using the fact that the line integral of a complex function with an antiderivative is zero and the above, conclude that

   $$\int_C \frac{f'(z)}{z-z_0}\,dz = \int_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

   (c) Now apply Cauchy's formula to conclude that

   $$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz.$$