Proof Workshop

Contents

4 Week 4: Limits, Continuity, and Epsilon-Delta Proofs

Today, we are in a wonderland. This wonderland is $\mathbb{R}$.

4.1 Limits of Sequences

Definition 4.1 (Sequence)**.** A sequence in $\mathbb{R}$ is a function $a : \mathbb{N} \to \mathbb{R}$.

Remark 4.2. Purely for convenience, we will assume that $\mathbb{N}$ does not contain 0 today, so that our sequences start at 1.

More often than not, we will notate a sequence $a : \mathbb{N} \to \mathbb{R}$ not as a function, but as a family of real numbers indexed by $\mathbb{N}$. In particular, we denote a by $\{a_n\}_{n=1}^{\infty}$, where $a_n = a(n)$ for each $n \in \mathbb{N}$. When the entire sequence can be inferred from the first few terms, sometimes we will even write only the first few terms. For example, we might write $1, 1, \ldots$ for the constant sequence $a_n = 1$.

Example 4.3. Here are some examples of sequences in $\mathbb{R}$.

(1) $a_n = n$, which yields $1, 2, 3, 4, \ldots$

(2) $a_n = 1/n$, which yields $1, 1/2, 1/3, 1/4, \ldots$

(3) $a_n = (2^n - 1)/2^n$, which yields $1/2, 3/4, 7/8, 15/16, \ldots$

(4) $a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$, which yields $1, 2, 3, 5, 8, 13, \ldots$

(5) Let

$$a_n = \begin{cases} 2^n & n \text{ is odd} \\ n & n \text{ is even.} \end{cases}$$

Then a_n is the sequence $2, 2, 8, 4, 32, 6, \ldots$.

Remark 4.4. A sequence of real numbers and a set of real numbers are not the same thing. For instance, the sequence $1, 1/2, 1/4, 1/8, \ldots$ is different from the sequence $1/8, 1, 1/2, 4, \ldots$, even though the sets $\{1, 1/2, 1/4, 1/8, \ldots\}$, and $\{1/8, 1, 1/2, 4, \ldots\}$ are identical. For a sequence $\{a_n\}$, the set containing the sequence is written as $\{a_n : n \in \mathbb{N}\}$.

Something we might be curious about is how a sequence behaves in the long run, i.e., as n tends to ∞. After all, a sequence can do any number of things. It might converge to a finite value, it might tend to one extreme or another, or it might oscillate indefinitely.

Definition 4.5 (Limit of a Sequence)**.** We say that L is the **limit** of the sequence $\{a_n\}_{n=1}^{\infty}$ as n tends to ∞, or equivalently,

$$\lim_{n\to\infty} a_n = L,$$

if for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ whenever $n \geqslant N$. In somewhat contrived mathematical notation, L is the limit of a_n if

$$\forall \varepsilon > 0 : (\exists N \in \mathbb{N} : (\forall n \geqslant N : |a_n - L| < \varepsilon))$$

The Limit Cookbook, for Sequences. Suppose you have a solid guess $L \in \mathbb{R}$ for what the limit of the function representing a sequence is. Proving that your guess is correct isn't difficult! Just follow the recipe below:

- Let $\varepsilon > 0$ be given.
- Conjure up a suitable N. This N usually depends on ε in some way.
- Verify that $|a_n - L| < \varepsilon$ for all $n \geqslant N$.

Along the way, you might find the following theorem useful.

Theorem 1 (Archimedean properties)

The following statements are equivalent and true.

(i) If a and b are real numbers and $a > 0$, then there exists an $n > 0$ such that $na > b$.

(ii) For every $x \in \mathbb{R}$, there exists an n such that $n \leqslant x < n + 1$.

(iii) For every $x > 0$, there exists $n > 0$ such that $1/n \leqslant x$.

Example 4.6. Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof. Let $\varepsilon > 0$. Let $N = \lfloor \frac{1}{\varepsilon} \rfloor + 1$. Then for all $n > N$,

$$|a_n - L| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

Why did we choose this δ? The equality $\frac{1}{N} = \varepsilon$ has solution $N = \frac{1}{\varepsilon}$. But we need this to be an integer, so we take the floor of this number. Now $\lfloor x \rfloor \leqslant x$ for all x, and so with $N = \lfloor \frac{1}{\varepsilon} \rfloor$ we have $\frac{1}{N} \geqslant \varepsilon$! To turn this into a strict inequality, we add a 1. □

Example 4.7. Show $\lim_{n\to\infty} \frac{4n+1}{n+3} = 4$.

Proof. Work backwards: $|x_n - 4| = |\frac{4n+1}{n+3} - 4| = |\frac{-11}{n+3}|$. Therefore, as with last time we set $N = \lfloor \frac{11}{\varepsilon} - 3 \rfloor + 1$.

Now let $\varepsilon > 0$ and let $N = \lfloor \frac{11}{\varepsilon} - 3 \rfloor + 1$. Then

$$|x_n - 4| = \left| \frac{4n+1}{n+3} - 4 \right| = \left| \frac{-11}{n+3} \right| = \frac{11}{n+3} < \frac{11}{N+3} < \varepsilon.$$ □

4.2 Limits of Functions

We can also make sense of what it means for a function f to have a limit.

Definition 4.8 (Limit of a Function)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function, and $c \in \mathbb{R}$. We say that L is the **limit** of $f(x)$ as x tends to c, or equivalently,

$$\lim_{x\to c} f(x) = L,$$

if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$.

The Limit Cookbook, for Functions. It is likewise easy to describe a recipe for preparing the limit L for $f(x)$:

- Let $\varepsilon > 0$ be given.
- Conjure up a suitable δ. This δ should probably depend on ε in some way.
- Verify that $|f(x) - L| < \varepsilon$ for any x such that $0 < |x - a| < \delta$.

When constructing the epsilon-delta proof, we need to determine the value for delta. To determine delta, it is helpful to begin with the final statement and work backwards. However, usually when we write up the proof, we simply present the delta found and show that it works without explanation of how we found it.

Of course the limit of an arbitrary function f at a point c need not exist (as the following exercise demonstrates).

Example 4.9. Show that $\lim_{x\to 4}(5x-7) = 13$.

Proof. We first work backwards. Let ε be given. We want to determine δ. We have

$$\begin{aligned} |f(x) - L| < \varepsilon &\implies |(5x-7) - 13| < \varepsilon \implies |5x - 20| < \varepsilon \\ &\implies |5||(x-4)| < \varepsilon \implies |x - 4| < \frac{\varepsilon}{5} \end{aligned}$$

Note that we want to now let $\delta = \frac{\varepsilon}{5}$. We can now write the proof:

Suppose $\varepsilon > 0$ has been provided. Define $\delta = \frac{\varepsilon}{5}$. Since $\varepsilon > 0$, we also have $\delta > 0$. Now for every x, the statement $0 < |x - c| < \delta$ implies $0|x - c| < \frac{\varepsilon}{5}$. Then

$$|5x - 20| < \varepsilon \implies |(5x-7) - 13| < \varepsilon.$$

Therefore $\lim_{x\to 4}(5x-7) = 13$. □

Example 4.10. Prove that $\lim_{x\to 5}(3x^2 - 1) = 74$.

Proof. Let's begin with some scratch work. Recall that the statement $|a| < b$ is equivalent to $-b < a < b$.

$$\begin{aligned} |f(x) - L| < \varepsilon &\implies |(3x^2 - 1 - 74| < \varepsilon \implies |3x^2 - 75| < \varepsilon \\ &\implies -\varepsilon < 3x^2 - 75 < \varepsilon \implies 25 - \frac{\varepsilon}{3} < x^2 < 25 + \frac{\varepsilon}{3}. \end{aligned}$$

Since the square root function is increasing, it preserves $<$. So

$$\sqrt{25 - \frac{\varepsilon}{3}} < x < \sqrt{25 + \frac{\varepsilon}{3}} \implies -5 + \sqrt{25 - \frac{\varepsilon}{3}} < x - 5 < -5 + \sqrt{25 + \frac{\varepsilon}{3}}.$$

where we subtracted 5 since we want to evaluate the limit there. There are now two candidates for δ, and δ needs to be less than or equal to both of them. We can just let

$$\delta = \min\left\{\sqrt{25 - \frac{\varepsilon}{3}}, -5 + \sqrt{25 + \frac{\varepsilon}{3}}\right\}.$$

However, note that the expression on the left is undefined for $\varepsilon > 75$. We will handle this situation by introducing a smaller ε in the proof.

We can now prove the limit. Suppose we are given $\varepsilon > 0$. Let $\varepsilon_2 = \min\{\varepsilon, 72\}$ (this avoids the "large ε" situation). Define

$$\delta = \min\left\{\sqrt{25 - \frac{\varepsilon_2}{3}}, -5 + \sqrt{25 + \frac{\varepsilon_2}{3}}\right\}.$$

Since $\varepsilon_2 > 0$, we also have $\delta > 0$. Now for every x, the expression $0 < |x - c| < \delta$ implies

$$-\delta < x - c < \delta \implies -5 + \sqrt{25 - \frac{\varepsilon_2}{3}} < x - 5 < -5 + \sqrt{25 + \frac{\varepsilon_2}{3}}$$

which we have seen is equivalent to

$$-\varepsilon_2 < 3x^2 - 75 < \varepsilon_2.$$

Therefore

$$|3x^2 - 75| = |(3x^2 - 1) - 74| < \varepsilon_2 \leqslant \varepsilon.$$

Therefore, $\lim_{x\to 5}(3x^2 - 1) = 74$. □

4.3 Continuity

Definition 4.11 (Continuous function)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **continuous** at $a \in \mathbb{R}$ if

$$\lim_{x\to a} f(x) = f(a).$$

The same function f is said to be *continuous* if f is continuous at a for every $a \in \mathbb{R}$.

Example 4.12. For all $p \in \mathbb{R}$, $p > 0$, $\lim_{x\to p} \sqrt{x} = \sqrt{p}$.

Proof. Given $\varepsilon > 0$ we must show that $|\sqrt{x} - \sqrt{p}| < \varepsilon$, provided that x and p are close enough. Now

$$\left|\sqrt{x} - \sqrt{p}\right| = \frac{|x-p|}{|\sqrt{x} + \sqrt{p}|} < \frac{|x-p|}{\sqrt{p}}.$$

Therefore, choosing $\delta = \frac{\varepsilon}{\sqrt{p}}$ gives the desired result. □

Example 4.13. If $f, g : \mathbb{R} \to \mathbb{R}$ are continuous at a, then $f + g$ is continuous at a.

Proof. Fix some $\varepsilon > 0$. We want to produce a $\delta > 0$ such that

$$|x - a| < \delta \implies |(f(x) + g(x)) - (f(a) + g(a))| < \varepsilon.$$

Let us look at ways we can bound this second expression. A first trick to always try is using the triangle inequality, that is, the fact that $|a+b| \leqslant |a| + |b|$. Rearranging terms in our earlier expression, we get

$$|f(x) + g(x) - f(a) - g(a)| = |f(x) - f(a) + g(x) - g(a)| \leqslant |f(x) - f(a)| + |g(x) - g(a)|.$$

The use in rearranging this way is that it allows us to take advantage of the continuity of f and g at a. For any $\varepsilon > 0$, we know there exist $\delta_1, \delta_2 > 0$ such that

$$|x - a| < \delta_1 \implies |f(x) - f(a)| < \varepsilon,$$

$$|x - a| < \delta_2 \implies |g(x) - g(a)| < \varepsilon.$$

Then when $|x - a|$ is less than both δ_1 and δ_2, we have that

$$|f(x) + g(x) - f(a) - g(a)| \leqslant |f(x) - f(a)| + |g(x) - g(a)| < \varepsilon + \varepsilon = 2\varepsilon.$$

Therefore, to bound $|f(x) + g(x) - f(a) - g(a)|$ by ε, we need to bound $|x - a|$ by the δ_1, δ_2 making $|f(x) - f(a)|, |g(x) - g(a)|$ less than $\varepsilon/2$. We know this is possible, again by continuity. It is then easy to check that if $\delta = \min(\delta_1, \delta_2)$, then we get $|f(x) + g(x) - (f(a) + g(a))| < \varepsilon$. □

Two key takeaways here:

1. Use the triangle inequality! It is extremely useful and common in epsilon-delta proofs, as it allows you to split large sums in absolute values into more 'bite-sized,' workable pieces.

2. If you are given the continuity of some function, and then that function is used to make another function (as $f + g$ was in this example), your new function will generally 'inherit' its δ from some δ of the function you know to be continuous. See how much the continuity of the function you are given can buy you.

In the exercises, you will have the opportunity to explore practice using ε-δ proofs on basic functions, prove some properties of limits, and explore equivalent definitions of continuity as well as what it means for a sequence to converge.