FREE OBJECTS AND TENSOR PRODUCTS OF MODULES OVER A COMMUTATIVE RING

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If you have taken a standard abstract algebra course, then you have probably heard of free groups. But, most such courses do not introduce the reader to the language of category theory, which unifies the notion of a free object. In the present lecture, we will define a free group categorically, and then go on to define a free module over a commutative ring, and hence, a free abelian group (which is just a $\mathbb{Z}$–module).

For present purposes, we will assume that free groups exist (if you are curious, just pick up any standard abstract algebra textbook and take a look at the construction of a free group).

Definition: Given a category $\mathcal{C}$, $i \in ob\mathcal{C}$ is called an *initial object* iff $\forall a \in ob\mathcal{C}$, $\exists!$ morphism $\varphi : i \longrightarrow a$. Similarly, $f \in ob\mathcal{C}$ is called a *final (terminal) object* iff $\forall a \in ob\mathcal{C}$, $\exists!$ morphism $\phi : a \longrightarrow f$.

An object in a category $\mathcal{C}$ is called a *universal object* iff it is initial or final in $\mathcal{C}$.
Since, initial and final objects in a category are unique up to unique isomorphisms, hence, universal objects are unique up to unique isomorphisms.

Free Groups:

Given any set A, the goal is define a group G which contains A in the most efficient way. This vague notion is formalized using category theory as follows:

Define a category $\mathcal{C}_A$ whose objects are ordered pairs (G, f) where G is a group and $f : A \longrightarrow G$ is a set theoretic function. Given any two objects (G_1, f_1) and (G_2, f_2), a morphism $\varphi \in Hom((G_1, f_1), (G_2, f_2))$ is a group homomorphism $\varphi : G_1 \longrightarrow G_2$ such that $\varphi \circ f_1 = f_2$, i.e., the following diagram commutes:

$$\begin{tikzcd} G_1 \arrow[rr, dashed, "\varphi"] & & G_2 \\ & A \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

*Fig*1

A *free group*, by definition, is an initial object in $\mathcal{C}_A$. It is denoted as $(\mathcal{F}(A), \pi)$, and a further simplification is done by dropping the function π, and denoting the free group by just $\mathcal{F}(A)$.

Explicitly, given a set A, a free group on A, is a group $\mathcal{F}(A)$ together with a function $\pi : A \longrightarrow \mathcal{F}(A)$, satisfying the following universal property (called *the universal property of a free group*): given any group G and a function $f : A \longrightarrow G$, $\exists!$ group homomorphism $\varphi : \mathcal{F}(A) \longrightarrow G$, such that $\varphi \circ \pi = f$, i.e., the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(A) \arrow[rr, dashed, "\exists!\varphi"] & & G \\ & A \arrow[ul, "\pi"] \arrow[ur, "f"'] & \end{tikzcd}$$

*Fig*2

Since, a free group $\mathcal{F}(A)$ on a set A is a universal object, hence, it is unique up to a unique isomorphism. Thus, the notation $(\mathcal{F}(A), \pi)$ is not ambiguous, and we can talk about $'the'$ free group on a set A. As an exercise one can try to show that the usual construction of a free group (using words out of elements of A and juxtaposition as binary operation) satisfies the universal property of a free group stated above (what is the map $\pi : A \longrightarrow \mathcal{F}(A)$?).

Free Modules:

Let R be a commutative ring. Given a set M (remember, a set not a module), our goal is to define an R–module that contains M in the most efficient way. The construction of a free R–module on the set M parallels the construction of a free group.

Define a category $\mathcal{C}_M$ whose objects are ordered pairs (N, f) where N is an R–module, and $f : M \longrightarrow N$ is a set theoretic function. Given any two objects (N_1, f_1) and (N_2, f_2), a morphism $\varphi \in Hom((N_1, f_1), (N_2, f_2))$ is an R–module homomorphism $\varphi : N_1 \longrightarrow N_2$ such that $\varphi \circ\ f_1 = f_2$, i.e., the following diagram commutes:

$$\begin{tikzcd} N_1 \arrow[rr, dashed, "\varphi"] & & N_2 \\ & M \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

*Fig*3

A *free R–module*, by definition, is an initial object in $\mathcal{C}_M$. It is denoted as $(\mathcal{F}(M)_R, \pi)$. Usually the map π is omitted and the free module is just denoted as $\mathcal{F}(M)_R$.

Explicitly, given a set M, a free R–module on M, is a module $\mathcal{F}(M)_R$ together with a function $\pi : M \longrightarrow \mathcal{F}(M)_R$, satisfying the following universal property (called *the universal property of a free R–module*): given an R–module N and a function $f : M \longrightarrow N$, $\exists!$ R–module homomorphism $\varphi : \mathcal{F}(M)_R \longrightarrow N$, such that $\varphi \circ \pi = f$, i.e., the following diagram commutes:

$$
\begin{tikzcd}
\mathcal{F}(M)_R \arrow[rr, dashed, "\exists!\varphi"] & & N \\
& M \arrow[ul, "\pi"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

*Fig*4

Since a free R–module $\mathcal{F}(M)_R$ on a set M is a universal object, hence, if it exists, it is unique up to a unique isomorphism. Thus, the notation $(\mathcal{F}(M)_R, \pi)$ is not ambiguous. It remains to show existence. But, we will have to define a few things first, to keep this lecture self-contained. Most of you probably know the next few definitions from linear algebra.

Definition: Given an R–module N, and $S \subset N$, the *module generated by S*, denoted $< S >$ is the intersection of all submodules of N (note that an arbitrary intersection of submodules of N is again a submodule of N) containing S. Thus, it is the smallest (w.r.t. $\subset$) submodule of N containing S.

Exercise: Show that $< S > = \{ \sum_{1 \leq i \leq n} r_i s_i : n \in \mathbb{N}, r_i \in R, s_i \in S\}$.

Definition:
• Given an R–module N, we say that N is *finitely generated* iff $\exists S \subset N$ such that S is a finite set, and $N =< S >$.
• A subset $S \subset N$ is *linearly independent* iff $\forall n \in \mathbb{N}$, $\sum_{1 \leq i \leq n} r_i s_i = 0 \Rightarrow \forall i \in \{1, ..., n\}, r_i = 0$.
• S is a *basis* for N iff $N =< S >$ and S is linearly independent.

We are now in a position to show the existence of a free module.

Proposition: Given a ring R, and a set M, a free R–module on M exists.

Proof: Recall that R can be regarded as an R–module, and so, one can form the direct sum $\bigoplus_{m \in M} R_m$, where $\forall m \in M, R_m = R$. The direct sum $\bigoplus_{m \in M} R_m$ is also denoted as $R^{\bigoplus M}$. We will show that $R^{\bigoplus M}$ satisfies the universal property of a free module.

First, define $\pi : M \longrightarrow R^{\bigoplus M}$ as follows: $\forall m \in M, \pi(m) = (r_{i,m})_{i \in M}$ where $r_{i,m} = \delta_{i,m}$. It can be easily shown that $\{(r_{i,m})_{i \in M} : m \in M\}$ is a basis for $R^{\bigoplus M}$. Denote $(r_{i,m})_{i \in M}$

as e_m. Then every element of $R^{\bigoplus M}$ can be uniquely expressed as a linear combination $\sum_{m \in M} r_m e_m$ where $r_m \in R$ (note that a linear combination is always a finite sum, so that all but only a finitely many of the r_m are zero).

Now, let N be an R–module and $f : M \longrightarrow N$ be any set function. Define $\varphi : R^{\bigoplus M} \to N$ as follows: $\forall \sum_{m \in M} r_m e_m \in R^{\bigoplus M}, \varphi\left(\sum_{m \in M} r_m e_m\right) := \sum_{m \in M} r_m f(m)$. One can easily verify that φ is R–linear, and, moreover, $\varphi \circ \pi = f$.

Thus, it suffices to show that φ is unique. So, let $\phi : R^{\bigoplus M} \to N$ be an R–linear map such that $\phi \circ \pi = f$. Then, $\forall m \in M, \phi \circ \pi(m) = f(m) \Rightarrow \forall m \in M, \phi(\pi(m)) = f(m) \Rightarrow \forall m \in M, \phi(e_m) = f(m)$. From the linearity of ϕ it easily follows that $\phi = \varphi$, proving that φ is unique. Thus, by the universal property of a free module, $R^{\bigoplus M}$ is a free R–module on M.

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Since, free modules are unique up to unique isomorphisms, hence, one can define (in a less abstract manner) a free module to be an R–module N such that $N \approxeq R^{\bigoplus M}$ for some set M. This is the definition given in *Atiyah- Macdonald*. By convention if $M = \emptyset$ then $R^{\bigoplus M} = \{0\}$, the zero/ trivial module.

Exercise: Prove that an R–module N is free iff it has a basis (Hint: Use the less abstract definition of a free module given above).

Thus, all vector spaces over a field k are free k–modules.

Definition: Since abelian groups are $\mathbb{Z}$–modules, a *free abelian group* is just a free $\mathbb{Z}$–module, i.e., a *free abelian group on a set S* is just an abelian group that is isomorphic to $\mathbb{Z}^{\bigoplus S}$.

Given a ring R and a set M, $R^{\bigoplus M}$ is equipped with the standard basis $\{e_m : m \in M\}$, where e_m is as defined in the previous proposition. Then $R^{\bigoplus M} = \{\sum_{m \in M} r_m e_m : r_m \in R$ and finitely many r_m are non-zero$\}$.

Tensor Products:

Given the category *R-mod*, we have a binary operation $\bigoplus :$ *ob(R-mod)* $\times$ *ob(R-mod)* $\longrightarrow$ *ob(R-mod)* given by the direct sum of two R–modules. We want to define another binary operation $\bigotimes :$ *ob(R-mod)* $\times$ *ob(R-mod)* $\longrightarrow$ *ob(R-mod)*, which one may think of intuitively as a way of multiplying two R–modules to get another R–module. Our goal is to define this binary operation $\bigotimes$ which we call the *tensor product*.

Definition: Given R–modules M, N, P, a map $\mu : M \times N \longrightarrow P$ is called an *R–bilinear map* iff $\forall m, m' \in M, n, n' \in N, r \in R$:

- $\mu(m + m', n) = \mu(m, n) + \mu(m', n)$
- $\mu(m, n + n') = \mu(m, n) + \mu(m, n')$
- $\mu(rm, n) = r\mu(m, n) = \mu(m, rn)$.

Thus, μ is linear on both coordinates.

Examples:

(1) Given a ring R, let $\bullet : R \times R \longrightarrow R$ denote multiplication in R. Then $\bullet$ is R–bilinear.

(2) Given an R–module N, if R is regarded as an R–module, then the module action from $R \times M \longrightarrow M$ is R–bilinear.

(3) For $n \in \mathbb{N}$, let $\mathbb{R}^n$ denote the n-tuples of real numbers with the usual $\mathbb{R}$–vector space structure. Let $\bullet : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ denote the standard dot product. Then $\bullet$ is an $\mathbb{R}$–bilinear map.

The construction of a tensor product of two modules is quite similar to the construction of free groups and free modules we have seen so far, but has important differences as well.

Definition: Let M, N be R–modules. Define a category $\mathcal{C}_{M,N}$ whose objects are ordered pairs (P, φ) where P is an R–module, and $\varphi : M \times N \longrightarrow P$ is an R–bilinear map. Given any two objects $(P_1, \varphi_1), (P_2, \varphi_2)$, a morphism $\phi \in Hom((P_1, \varphi_1), (P_2, \varphi_2))$ is an R–module homomorphism from $P_1 \longrightarrow P_2$ such that $\phi \circ \varphi_1 = \varphi_2$, i.e., the following diagram commutes:

$$\begin{tikzcd}
P_1 \arrow[rr, dashed, "\phi"] & & P_2 \\
& M \times N \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

$Fig5$

A *tensor product* of M, N is an initial object in $\mathcal{C}_{M,N}$. It is usually denoted as $(M \bigotimes_R N, \pi)$. Note that being a universal object, a tensor product of M, N (if it exists) is unique up to a unique isomorphism. Thus, the notation $(M \bigotimes_R N, \pi)$ is not ambiguous. The R–bilinear map π is usually suppressed, and the tensor product is denoted as $M \bigotimes_R N$. It is essential to check whether $\mathcal{C}_{M,N}$, as defined, has an initial object. Otherwise, this entire construction is not very useful, for our goal is to define a binary operation $\bigotimes$: *ob(R-mod)* $\times$ *ob(R-mod)* $\longrightarrow$ *ob(R-mod)*. Now, our knowledge of free modules will come in handy. So, learning about them was not a complete waste of time!

Proposition: (We will use the result in this proposition when we show the existence of a tensor product) Given R–modules M, N, let $\varphi : M \longrightarrow N$ be an R–linear map. If $S \subset M$ be a submodule such that $S \subset ker\varphi$, then given the standard projection map $\pi : M \longrightarrow \frac{M}{S}$, $\exists!$ R–linear map $\varphi^* : \frac{M}{S} \longrightarrow N$ such that $\varphi^* \circ \pi = \varphi$, i.e., the following diagram commutes:

$$\begin{tikzcd}
\frac{M}{S} \arrow[rr, dashed, "\varphi^*"] & & N \\
& M \arrow[ul, "\pi"] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

*Fig*6

Proof: The proof of this proposition will be omitted for the purposes of this lecture. It is quite easy, and can be read in any standard abstract algebra book.

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Proposition: Given R–modules M, N, there exists an R–module T and an R–bilinear map $\pi : M \times N \longrightarrow T$ such that if P is any R–module and $\varphi : M \times N \longrightarrow P$ is an R–bilinear map, then $\exists!$ R–linear map $\phi : T \longrightarrow P$ such that $\phi \circ \pi = \varphi$.

Proof: Consider the free R–module $R^{\bigoplus M \times N}$ on the set $M \times N$. Then, as mentioned before, $R^{\bigoplus M \times N} = \{ \sum_{(m,n) \in M \times N} r_{(m,n)} e_{(m,n)} : r_{(m,n)} \in R$ and finitely many $r_{(m,n)}$ are non-zero$\}$. We have a natural projection map (just a function) $p_1 : M \times N \longrightarrow R^{\bigoplus M \times N}$ given by $p_1(m, n) = e_{(m,n)}$.

For all $(m, n) \in M \times N$, $r \in R$, let $S \subset$(m, n) $\in M \times N$, $r \in R$ be the submodule of $R^{\bigoplus M \times N}$ generated by all elements of the form:

- ⋆ $e_{(m+m',n)} - e_{(m,n)} - e_{(m',n)}$
- ⋆ $e_{(m,n+n')} - e_{(m,n)} - e_{(m,n')}$
- ⋆ $e_{(rm,n)} - re_{(m,n)}$
- ⋆ $e_{(m,rn)} - re_{(m,n)}$

Define $T := \dfrac{R^{\bigoplus M \times N}}{S}$. Then T is clearly an R–module. We have a natural R–linear projection map $p_2 : R^{\bigoplus M \times N} \longrightarrow T$ given by $p_2 \left(\sum_{(m,n) \in M \times N} r_{(m,n)} e_{(m,n)} \right) := \sum_{(m,n) \in M \times N} r_{(m,n)} e_{(m,n)} + S$. Define $\pi := p_2 \circ p_1$. It is easy to verify that $\pi : M \times N \longrightarrow T$ is R–bilinear (use the definition of T). For all $(m, n) \in M \times N, \pi(m, n) = e_{(m,n)} + S$ is denoted as $m \otimes n$. T is generated as an R–module by elements of the form $m \otimes n$ for $m \in M, n \in N$.

We have a map $p_1 : M \times N \longrightarrow R^{\bigoplus M\times N}$, and a map $\varphi : M \times N \longrightarrow P$. Thus, by the universal property of the free product $R^{\bigoplus M\times N}$, $\exists!$ R–module homomorphism $\phi : R^{\bigoplus M\times N} \longrightarrow P$, such that $\phi \circ p_1 = \varphi$, i.e., the following diagram commutes:

$$\begin{tikzcd}
R^{\bigoplus M\times N} \arrow[r, dashed, "\exists!\phi"] & P \\
M \times N \arrow[u, "p_1"] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

*Fig*7

It should be clear that $\phi\left(\sum_{(m,n)\in M\times N} r_{(m,n)}e_{(m,n)}\right) = \sum_{(m,n)\in M\times N} r_{(m,n)}\varphi(m,n)$. We will show that $S \subset ker\phi$. Clearly, $\phi(e_{(m+m',n)} - e_{(m,n)} - e_{(m',n)}) = \phi(e_{(m+m',n)}) - \phi(e_{(m,n)}) - \phi(e_{(m',n)}) = \phi(p_1(m+m',n)) - \phi(p_1(m,n)) - \phi(p_1(m',n)) = \varphi(m+m',n) - \varphi(m,n) - \varphi(m',n) = 0$ (Since, φ is bilinear). It can be similarly show that ϕ vanishes on every other generator of S, and so ϕ vanishes on the whole of S, i.e., $S \subset ker\phi$. Thus, by the proposition stated before, $\exists!$ R–module homomorphism $\phi^* : T \longrightarrow P$ such that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[r, dashed, "\exists!\phi^*"] & P \\
R^{\bigoplus M\times N} \arrow[u, "p_2"] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

*Fig*8

Thus, the following diagram commutes:

$$\begin{tikzcd}
T \arrow[dr, "\phi^*"] & \\
 & P \\
M \times N \arrow[uu, "\pi"] \arrow[ur, "\varphi"] &
\end{tikzcd}$$

*Fig*9

It suffices to show that ϕ^* is the unique R–linear map that makes $Fig9$ commute. So, let $\gamma : T \longrightarrow P$ be an R–linear map such that that $\gamma \circ \pi = \varphi$. Then $\gamma \circ (p_2 \circ p_1) = \varphi \Rightarrow (\gamma \circ p_2) \circ p_1 = \varphi$. Thus, $\forall (m,n) \in M \times N, (\gamma \circ p_2) \circ p_1(m,n) = \varphi(m,n) \Rightarrow \forall (m,n) \in M \times N, \gamma \circ p_2(e_{(m,n)}) = \varphi(m,n)$. Since, γ, p_2 are R–linear, hence $\gamma \circ p_2$ is R–linear. So,

$$\gamma \circ p_2 \left(\sum_{(m,n) \in M \times N} r_{(m,n)} e_{(m,n)} \right) = \sum_{(m,n) \in M \times N} r_{(m,n)} \varphi(m,n) = \phi \left(\sum_{(m,n) \in M \times N} r_{(m,n)} e_{(m,n)} \right).$$

By the uniqueness of ϕ^* it follows that $\gamma = \phi^*$, proving that ϕ^* is the unique R–linear map that makes $Fig9$ commute.

By the universal property of a tensor product, (T, ϕ^*) is a tensor product of the R–modules M, N.

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Now that we have shown that the tensor product of modules exist, we can actually forget the construction of the tensor product. We will work with tensor products by using the universal mapping property.