Lecture 1: Motivating abstract nonsense

Nilay Kumar

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This summer, the UMS lectures will focus on the basics of category theory. Rather dry and substanceless on its own (at least at first), category theory is difficult to grok without proper motivation. With the proper motivation, however, category theory becomes a powerful language that can concisely express and abstract away the relationships between various mathematical ideas and idioms. This is not to say that it is purely a tool – there are many who study category theory as a field of mathematics with intrinsic interest, but this is beyond the scope of our lectures.

Functoriality

Let us start with some well-known mathematical examples that are unrelated in content but similar in "form". We will then make this rather vague similarity more precise using the language of categories.

Example 1 (Galois theory)**.**
Recall from algebra the notion of a finite, Galois field extension K/F and its associated Galois group $G = \mathrm{Gal}(K/F)$. The fundamental theorem of Galois theory tells us that there is a bijection between subfields $E \subset K$ containing F and the subgroups H of G. The correspondence sends E to all the elements of G fixing E, and conversely sends H to the fixed field of H. Hence we draw the following diagram:

$$\begin{tikzcd} K \arrow[r, leftrightsquigarrow] & 0 \arrow[d, hook] \\ E \arrow[u, hook] \arrow[r, leftrightsquigarrow] & H \arrow[d, hook] \\ F \arrow[u, hook] \arrow[r, leftrightsquigarrow] & G \end{tikzcd}$$

I denote the Galois correspondence by squiggly arrows instead of the usual arrows. Unlike most bijections we usually see between say elements of two sets (or groups, vector spaces, etc.), the fundamental theorem gives us a one-to-one correspondence between two very different types of objects: fields and groups. This is precisely why Galois theory is so useful; we can reduce difficult questions about, say, solvability by radicals of polynomials over a field, to possibly simpler questions about the solvability of a group, *without losing any structure.*

Example 2 (The fundamental group)**.**
In topology it is often useful to be able to distinguish between various topological spaces. This drives the study of topological invariants: data associated to homeomorphism (or homotopy) classes of spaces.

One of the simplest examples is the fundamental group $\pi_1(X, x_0)$, the group of all "homotopy classes of loops" in X based at x_0. A loop is a continuous map $\gamma : S^1 \to X$, and two loops γ, δ are said to be homotopic if there is a continuous map $F : S^1 \times [0,1] \to X$ such that $F(x,0) = \gamma$ and $F(x,1) = \delta$. Hence, as a set, $\pi_1(X, x_0)$ is the set of all loops in X upto continuous deformation. Indeed, $\pi_1(X, x_0)$ can be given a group structure. The product of two loops is the loops obtained by traversing each in turn, and so the identity is the loop given by $S^1 \mapsto \{pt\}$. Let's look at a few examples.

Let $X = \mathbb{R}^n$. It's pretty clear that for any choice of basepoint $x_0 \in X$ any two loops can be continuously deformed to each other, i.e. there is only one homotopy class of loops. Thus, $\pi_1(X, x_0) = 0$, the trivial group. The first non-trivial example is the circle, S^1. Intuitively, one sees that a loop wrapping around the circle once cannot be continuously deformed to a loop wrapping around the circle twice. And indeed, it can be shown (although it takes some work) that $\pi_1(S^1, x_0) \cong \mathbb{Z}$. With this result in hand, it is not hard to compute fundamental groups of other run-of-the-mill spaces.

This formalism is quite useful; it is easy to see that if $X \cong Y$ are homeomorphic spaces then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ (where x_0 is identified with y_0).[1] Thus any two spaces with different fundamental groups cannot be homeomorphic! I want to pause here for a moment and note that we have assigned to each topological space a group. This assignment is in fact well-defined on homeomorphism classes of spaces, which is why the fundamental group is useful as an invariant. Hence we reduce a topological problem of distinguishing two spaces to a problem of computing (and comparing) their fundamental groups, a task that can often be reduced to algebra. Compare this to the case of Galois theory above: the fundamental group, like the fundamental theorem of Galois theory, associates objects from two very different worlds ("categories") in some structure-preserving ("functorial") way that turns out to be quite useful.

Let us now make the similarity between the two examples above more precise. We define a **category** to consist of "objects" and "morphisms," where every object has a distinguished identity morphism, and morphisms can be composed in the usual way. We've already seen a few examples: there is the category **Top** with topological spaces as objects and continuous maps as morphisms, the category **Grp** with groups as objects and group homomorphisms as morphisms, and the category **Field** with fields as objects and ring homomorphisms (i.e. field extensions) as morphisms. Even simpler is the category **Set** with (you guessed it) sets as objects and set maps as morphisms.

But the idea of a category is old hat to the point where it seems almost trivial – we're used to restricting our domain of objects of interest to the objects of a single category. What's actually interesting about the above examples can be neatly encapsulated in the following definition. A **functor** $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a map that takes every object $c \in \mathcal{C}$ to an object $F(c)$ in $\mathcal{D}$ *and* every morphism $\alpha : c_1 \to c_2$ of objects $c_1, c_2 \in \mathcal{C}$ to a morphism $F\alpha : F(c_1) \to F(c_2)$ in $\mathcal{D}$.

In this sense, the fundamental theorem of Galois theory provides us with two functors: let K/F be Galois. Let $\mathcal{L}$ be the category whose objects are intermediate fields between F and K and whose morphisms are field extensions. Let $\mathcal{G}$ be the category whose objects are subgroups of $G = \mathrm{Gal}(K/F)$ and morphisms are inclusions. Then there the fundamental theorem provides us with a functor $S : \mathcal{L} \to \mathcal{G}$ taking a field extension E of F to the Galois group $\mathrm{Gal}(K/E)$ and taking the inclusion $F \hookrightarrow E$ to the inclusion $\mathrm{Gal}(K/E) \hookrightarrow \mathrm{Gal}(K/F)$. Moreover, we have a functor $S' : \mathcal{G} \to \mathcal{L}$ taking a subgroup $H \subset G$ to the fixed field K^H and the inclusion $H \hookrightarrow G$ to the inclusion $F \hookrightarrow K^H$. Note that these functors are inclusion-reversing – we call such functors

[1] In fact, more is true. If X and Y are only homotopy equivalent, i.e. there exist $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to Id_Y, then they have the same fundamental group.

contravariant. The two functors S and S' in this case are "inverses" in a sense (that we will not elaborate on today), which gives Galois theory its power.

Now consider the fundamental group. It takes a topological space X together with a fixed point $x_0 \in X$ to a group. Consider the pair (X, x_0), a so-called pointed topological space. We write the category of pointed topological spaces (together with continuous maps $f : (X, x_0) \to (Y, y_0)$ with $f(x_0) = y_0$) cleverly as **Top.**. We thus think of the fundamental group as a functor $\pi_1 : \mathbf{Top.} \to \mathbf{Grp}$.[2] It is not hard to check that a continuous map $f : (X, x_0) \to (Y, y_0)$ induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Note that the order of domain/codomain is preserved, unlike in the case of Galois theory; we say that π_1 is a **covariant** functor. Note that unlike in the case of Galois theory, there is no functor going the other way, which is why the fundamental group is not a complete topological invariant: there exist non-homeomorphic spaces with isomorphic fundamental group.

Mathematical reuse

Before I wrap up this lecture, let me briefly discuss another nice feature of categorical language. Often when one is learning about or working with new, unfamiliar objects, it is not clear why certain constructions deserve the name that they do. Ravi Vakil offers the following example:

> For example, we will define the notion of *product* of schemes. We could just give a definition of product, but then you should want to know why this precise definition deserves the name of "product"... We will be creating some new mathematical objects (such as schemes, and certain kinds of sheaves), and we expect them to act like objects we have seen before.

The point is that category theory allows us to say precisely what it means to "act like" a product object, for example. Indeed, one can define a product of two objects in some category $\mathcal{C}$ via the following **universal property of the product**: an object X is the product of X_1 and X_2, denoted $X_1 \times X_2$, if and only if there exist morphisms $\pi_1 : X \to X_1$, $\pi_2 : X \to X_2$ such that for every object Y and pair of morphisms $f_1 : Y \to X_1, f_2 : Y \to X_2$, there exists a unique morphism such that the following diagram commutes:

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X_2 \end{tikzcd}$$

In line with the usual intuition of product objects, the morphism f is called the product of f_1 and f_2 and π_1, π_2 are called projection morphisms. You can check, in your favorite category, that this does indeed agree with the definition of product that you may be used to. The beautiful thing about such an abstract definition is that it captures the "idea" of what a product is. Because it's a definition via the "behavior" of an object, the universal property can be applied in any category.[3]

This example of using a universal property as a definition exemplifies the type of "reuse" that category theory affords. There are several, more drastic examples of reuse. Take, for example,

[2] As mentioned in the previous footnote, π cannot distinguish between homotopy equivalent spaces, and hence one can think of it as a functor $\pi_1 : \mathbf{Toph.} \to \mathbf{Grp}$ from homotopy types of pointed topological spaces to groups.

[3] It turns out that not all categories have products (consider, say, **Field**), but in those that do, the object defined by the universal property is the unique such object up to unique isomorphism. In other words, the universal property guarantees uniqueness but not existence. This is a general fact about universal properties that follows from some rather formal manipulation.

abelian categories. Such categories generalize the notion of the category **Ab** of abelian groups. Any abelian category has, among other things, a zero object, products and coproducts, and kernels and cokernels. This generalization is useful because many interesting categories (such as the category of modules over a ring or the category of sheaves of abelian groups on a topological space) are abelian: one can prove general results for an arbitrary abelian category, instead of rederiving such results in isolation when working in any given category. This turns out to be very convenient and quite powerful in algebraic topology and geometry; this is the subject of homological algebra.

Categories and beyond

There's a whole lot to category theory, as we will see in this seminar, but I hope that these examples give you an introductory overview of how category theory captures patterns and constructions that arise frequently in mathematics.

References

[1] Ravi Vakil, *Foundations of Algebraic Geometry.* June 11, 2013 version.