UMS 7/2/14

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Abstract

Notes for July, 2 2014 UMS lecture

1 Quick Review of Universals

Definition 1.1. If $S : D \to C$ is a functor and c an object of C, a universal arrow from c to S is a pair $\langle r, u\rangle$ consisting of an object r of D and an arrow $u : c \to Sr$ of C, such that to every pair $\langle d, f\rangle$ with d an object of D and $f : c \to Sd$ an arrow of C, there is a unique arrow $f' : r \to d$ of D with $Sf' \circ u = f$. In other words, every arrow f to S factors uniquely through the universal arrow u, as in the commutative diagram:

$$\begin{tikzcd} c \arrow[r, "u"'] & Sr \arrow[d, dotted, "Sf'"'] & r \arrow[d, dotted, "f'"'] \\ c \arrow[r, "f"'] & Sd & d \end{tikzcd}$$

This is the basic notion of a universal arrow from an object **to** a functor. The entire chapter will focus on different aspects of these universals. I find it best to think of these universals as a defining trait of the object and a corresponding functor that you can rely on, especially at this level of abstract nonsense. Maclane gives some good examples of universals on page 56, the bases of vector space one in particular being one that Kenny did last lecture. There is also a notion of a universal element, as follows:

Definition 1.2. If D is a category and $H : D \to \mathbf{Set}$ a functor, a universal element of the functor H is a pair $\langle r, e\rangle$ consisting of an object $r \in D$ and an element $e \in Hr$ such that for every pair $\langle d, x\rangle$ with $x \in Hd$, there is a unique arrow $f : r \to d$ of D with $(Hf)e = x$.

However, the notion of the universal element is just a special case of a universal arrow, as you can see discussed in the middle of page 58. Later, we will also see that universal elements and arrows subsume each other, so unless otherwise stated, we will work with universal arrows. Finally, we arrive at the dual concept of the universal arrow defined above.

Definition 1.3. A universal arrow from S to c is a pair $\langle r, v\rangle$ consisting of an object $r \in D$ and an arrow $v : Sr \to c$ with codomain c such that to every pair $\langle d, f\rangle$ with $f : Sd \to c$ there is a unique $f' : d \to r$ with $f = v \circ Sf'$, as in the commutative diagram:

$$\begin{tikzcd} d \arrow[d, dotted, "f'"'] & Sd \arrow[r, "f"'] \arrow[d, dotted, "Sf'"'] & c \\ r & Sr \arrow[r, "v"'] & c \end{tikzcd}$$

Notice that the difference between the examples is that one is **to** a functor, while the other is **from** a functor, a distinction that will allow us to define differences between limits and colimits later. There is a particularly simple example of universals on the bottom of page 58 which introduces the diagonal functor, a functor that will be important later. The kernel of a homomorphism in groups, for example, can also be seen as a universal for a suitable contravariant functor.

2 The Yoneda Lemma

This section will introduce some of the first non-trivial propositions, as well as some useful properties of universals. First, we discuss the notion of universality in terms of hom-sets, an often easier way of looking at things.

Proposition 2.1. *For a functor $S : D \to C$ a pair $\langle r, u : c \to Sr \rangle$ is universal from c to S if and only if the function sending each $f' : r \to d$ into $Sf' \circ u : c \to Sd$ is a bijection of him sets*

$$D(r, d) \cong C(c, Sd). \tag{1}$$

This bisection is natural in d. conversely, given r and c, any natural isomorphism as above is determined in this way by a unique arrow $u : c \to Sr$ such that $\langle r, u \rangle$ is universal from c to S.

The proof of the proposition can be seen on page 59, but for the dedicated student, we leave it as an exercise in definitions and diagram chasing. But from this proposition, we do get a new concept. If C and D have small hom-sets, then (1) tells us that the functor $C(c, S_{)}$ to **Set** is naturally isomorphic to the hom-functor $D(r,_{)}$ (It's important that they are small, and small in the categorical sense, or else it would not be a functor to the category **Set**. Ask yourself, why?) These isomorphisms are called representations:

Definition 2.2. Let D have small him-sets. A representation of a functor $K : D \to \mathbf{Set}$ is a pair $\langle r, \psi \rangle$, with r an object of D and

$$\psi : D(r,_{)} \cong K$$

a natural isomorphism. The object r is called the representing object. The functor K is said to be representable when such a representation exists.

Thus, representable functors are related up to isomorphism to the functor $D(r,_{)}$. How are these related to universal arrows? Well...

Proposition 2.3. *Let $*$ denote any one point set and let D have small him-sets. if $\langle r, u : * \to Kr \rangle$ is a universal arrow from $*$ to $K : D \to$ **Set**, then the function ψ which for each object d of D sends the arrow $f' : r \to d$ to $K(f')(u*) \in Kd$ is a representation of K. Every representation of K is obtained in this way from exactly one such universal arrow.*

Proof. For any set X, a function $f : * \to X$ from the one-point set $*$ to X is determined by the element $f(*) \in X$. Thus, this creates a bijection $\mathbf{Set}(*, K--) \to X$ given by $f \to f(*)$, natural in $X \in \mathbf{Set}$. If you compose with K, you get a natural isomorphism $\mathbf{Set}(*, K--) \to K$. But as seen in the definition of ψ, we get that

$$\mathbf{Set}(*, K--) \cong K \cong D(r, --)$$

So it suffices to find a natural isomorphism $\mathbf{Set}(*, K--) \cong D(r, --)$. But this follows from proposition 2.1. □

Therefore, we have that the notions universal element, universal arrow, and representable functor subsumes the other two. However, for those dedicated students who read the proof of proposition 2.1, there was a fact used, namely that each natural transformation of $\phi : D(r, --) \to K$ is completely determined but eh imaged under ψ_r of the identity $1_r : r \to r$. What follows is potentially the greatest lemma of all time.

Lemma 2.4. *(Yoneda) If $K : D \to$ **Set** is a functor from D and r an object in D (For D a category with small hom-sets), there is a bijection*

$$y : Nat(D(r, --), K) \cong Kr$$

which sends each natural transformation $\alpha : D(r, --) \to K$ to $\alpha_r 1_r$, the image of the identity $r \to r$.

The proof is evident by observing the following commutative diagram:

$$\begin{tikzcd}
D(r,r) \arrow[r, "\alpha_r"'] \arrow[d, "f*=D(r,f)"'] & K(r) \arrow[d, "K(f)"'] & r \arrow[d, "f"'] \\
D(r,d) \arrow[r, "\alpha_d"'] & K(d) & d
\end{tikzcd}$$

Corollary 2.5. *For objects $r, s \in D$, each natural transformation $D(r, --) \to D(s, --)$ hast the form $D(h, --)$ for a unique arrow $h : s \to r$.*

To Yoneda map, y, is a natural map, which is can be observed if we view K as an object in $\mathbf{Set}^D$, with both the domain and codomain of y as functors on the pair $\langle K, r\rangle$, which is an object in the category $\mathbf{Set}^D \times D$. The codomain of y would b eht evaluation functor E, which sends $\langle K, r\rangle$ to Kr, while the domain is the functor N which sends $\langle K, r\rangle$ to the set $Nat(D(r, --), K)$ of all natural transformations. Now we can add to the Yoneda Lemma.

Lemma 2.6. *The bijection y is a natural isomorphism $y : N \to E$ between the functors $E, N : \boldsymbol{Set}^D \times D \to \boldsymbol{Set}$.*

The object function $r \to D(r, --)$ and the arrow function

$$(f : s \to r) \mapsto d(f, --) : D(r, --) \to D(s, --)$$

for f an arrow of D together define a full and faithful functor

$$Y : D^{op} \to \mathbf{Set}^D$$

called the Yoneda functor. It's dual is another functor that is also faithful,

$$Y' : D \to \mathbf{Set}^{D^{op}}$$

. Make sure D has small hom-sets. If not, you can replace every **Set** in this section with **Ens** and the results will apparently still hold.

3 Coproducts and Colimits

This is where stuff gets funky. Let's begin with coproducts.

Definition 3.1. For any Category C, the diagonal functor $\Delta : C \to C \times C$ is defined by $\Delta(c) = \langle c, c\rangle, \Delta(f) = \langle f, f\rangle$, where c is an object and f is a function. A universal arrow from an object $\langle a, b\rangle$ of $C \times C$ to the functor Δ is called a *coproduct diagram.*

Basically, you need an object $c \in C$ and two arrows $i : a \to c, j : b \to c$. Of course, we must discuss the universal property: for any pair of arrows $f : a \to d, g : b \to d$ there is a unique $h : c \to d$ with $f = h \circ i, g = h \circ j$. When the diagram exists, we have c unique up to isomorphism, and we write $c = a \amalg b$ or $c = a + b$. c in this case is called the *coproduct* object. The universality can be seen in another diagram as follows:

$$\begin{tikzcd} a \arrow[r, "i"'] \arrow[dr, "f"'] & a \cup b \arrow[d, dotted, "h"'] & b \arrow[l, "j"'] \arrow[dl, "g"] \\ & d & \end{tikzcd}$$

Let's do an example. consider the category **Set**. Then take $a \amalg b$ to be the disjoint union of the sets a and b, while i and j are the inclusions maps of each respectively. Now a function h is determined by the composites hi and hj. This fulfills the diagram above. Disjoint unions are not necessarily unique, but they are up to a bijection, which befits a universal. Some more examples follow from these coproduct definitions in certain categories

$$\mathbf{Set} : thedisjointunionofsets$$

$$\mathbf{Top} : disjointunionofspaces$$

$$\mathbf{Group} : Freeproduct$$

$$Ab, \mathbb{R} - mod : directsum A \oplus B$$

and many more which I trust the reader to find on their own.

Infinite coproducts work as you would expect. Replace $C \times C = C^2$ with C^X where X is any set. Regarding the set S as a discrete category, the functor category C^X has as its objects the X-indexed family $a = \{a_x | x \in X\}$, and the diagonal functor Δ sends c to the constant family all $c_x = c$. The rest works as you would imagine, except there are multiple (infinite) injections from each C in the index. If you consider **Set**, then the infinite coproduct would be the X-fold disjoint union...

Copowers is just having all your a_x equal i.e. $a_x = b \forall x$. Now let's talk about more, of what you will soon learn, are special colimits...

Definition 3.2. Suppose that C has a null object z, such that for any two objects $b, c \in C$ there is a zero arrow $0 : b \to z \to c$. The *cockerel* of $f : a \to b$ is then an arrow $u : b \to e$ such that for

(i) $uf = 0 : a \to e$

(ii) if $h : b \to c$ has $hf = 0$, then $h = h'u$ for a unique arrow $h' : e \to c$.

The picture is as follows:
$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, "u"] \arrow[dr, "f'"] & e \arrow[d, dotted, "h'"] \\ & & d \end{tikzcd}$$

In **Ab**, the cokernel of $f : A \to B$ is the projection $B \to B/fA$. But what if we have no null object in our category. No problem. ABSTRACTION!

Definition 3.3. Given in C a pair $f, g : a \to b$ of arrows with the same domain a and the same codomain b, a *coequalizer* of $\langle f, g \rangle$ is an arrow $u : b \to e$ such that

(i) $uf = ug$

(ii) if $h : b \to c$ has $hf = hg$, then $h = h'u$ for a unique arrow $h' : e \to c$.

$$\begin{tikzcd} a \arrow[r, shift left, "f,g"] \arrow[r, shift right] & b \arrow[r, "u"] \arrow[dr, "f'"] & e \arrow[d, dotted, "h'"] \\ & & d \end{tikzcd}$$

You can also think of coequalizers as universal arrows! Denote $\downarrow\downarrow$ as the category with two objects and two non-identity arrows from the first object to the second. (Because of the difficulty of LateX, I will denotes this category as Q; the notation in Maclane is two dots with double right arrows between them.) Form the functor category $C^{\downarrow\downarrow}$ whose objects are functors from Q to C, i.e. a pair $\langle f, g \rangle : a \to b$ of parallel arrows. An arrow between pairs in this category is a natural transformation $\langle h, k \rangle$, with $h : a \to a'$ and $k : b \to b'$ in C. This makes the square diagram below commute: for both the f-square and add diagram g-square. One can also define a diagonal functor $\Delta : C \to C^{\downarrow\downarrow}$ by $\Delta(c) = \langle 1_c, 1_c \rangle, \Delta(r) = \langle r, r \rangle$, where c and r are objects and arrows of C respectively. It follows that a coequalizer of the pair $\langle f, g \rangle$ in $C^{\downarrow\downarrow}$ is the universal arrow from $\langle f, g, \rangle$ to the functor Δ. Now something new:

Definition 3.4. Given in C a pair $f : a \to b, g : a \to c$ of arrows with a common domain a, a *pushout* of $\langle f, g \rangle$ is a commutative square as shown on 66 and left
$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"] & b \arrow[d, "u"] \\ c \arrow[r, "v"] & r \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"] & b \arrow[d, "h"] \\ c \arrow[r, "k"] & s \end{tikzcd}$$
such that to every other commutative square similar to the right one, there is a $t : r \to s$ with $tu = h, tv = k$.

Basically, its the universal way to fill out a commutative square on the sides f,g. To view it as a universal, one must ask use the diagonal functor Δ which takes an object $c \in C$ to $\langle 1_c, 1_c \rangle$, i.e. $\Delta = \langle 1_c, 1_c \rangle$. I would advise reading page 66 for a more in-depth discussion on how this is a universal. Also, there is a section on the cokernel pair which I will not touch on greatly in these notes. The section is short so for the diligent student, I recommend reading it. For others, here is the basic definition:

Definition 3.5. Given an arrow $f : a \to b$ in C, the pushout of f with f is the cokernel pair of f. This is illustrated in the following diagram:
$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, shift left, "u,v"] \arrow[r, shift right] \arrow[dr, shift left, "h,k"] \arrow[dr, shift right] & r \arrow[d, dotted, "t"] \\ & & s \end{tikzcd} \quad uf = vf, hf = kf$$

If you haven't noticed, all these cases deal with particular functor categories, and can be generalized as such. Let C and J be categories(J for the index category). The diagonal functor

$$\Delta : C \to C^J$$

sends each object c to the constant functor Δc- the functor which has the value c at each object $i \in J$ and the value 1_c at each arrow of J. For an arrow $f : c \to c'$ of C, Δf is the natural transformation

$\Delta f : \Delta c \to \Delta c'$, which has the value of f at each object $i \in J$. Each functor $F : J \to C$ is an object of C^J by definition. Now a universal arrow $\langle r, u \rangle$ from F to Δ, with r an object of C and $u : F \to \Delta c$ a natural transformation, is called a colimit diagram for the functor F.

Definition 3.6. A universal arrow $\langle r, u \rangle$ from F to Δ, with r an object of C and $u : F \to \Delta c$ a natural transformation, is called a *colimit* for the functor F. We denote $r = ColimF$, and the natural transformation u to be universal among natural transformations $\tau : F \to \Delta c$. τ consists of arrows $\tau_i : F_i \to c, c \in C$ for each $i \in J$ and satisfies the property $\tau_j \circ Fu = \tau_i$ for each arrow $u : i \to j$:

$$\begin{tikzcd} F_1 \arrow[d, "\tau_1"] & F_2 \arrow[l, "Ff"'] \arrow[r, "Ff"] \arrow[d, "\tau_2"] & F_3 \arrow[d, "\tau_3"] \\ c & c & c \end{tikzcd}$$

(diagram keeps going for all $i \in J$, just wrote a portion of it)

This... is a crazy notion. But it's best not to question it and fuddle around: take it as it is and get familiar with it. This will be your bread and butter for the next few weeks. One can also identify the diagram of a colimit as a cone, by taking the vertex of the "cone" to be the common point c as in the diagram above. This results in the cone-like diagram:

$$\begin{tikzcd} F_i \arrow[r, "F_u"] \arrow[dr, "\tau_i"] & F_j \arrow[r, "Ff"] \arrow[d, "\tau_j"] & F_k \arrow[dl, "\tau_k"] \\ & c & \end{tikzcd}$$

A cone with the vertex $ColimF$ and the "cone" $\mu : F \to \Delta ColimF$ is called the *universal cone* or *limiting cone*. For any cone $\tau : F \to \Delta c$, one can find a unique arrow $t' : ColimF \to c$ such that $\tau_i = t' \mu_i$. Note that this c is different front the colimit of F, which is akin to the r defined above. I would advise going back through this section to make sure you have everything down, especially with which category each object and arrow is, like how $ColimF \in C$. It will help clear things up and make this large abstraction more comfortable.

4 Products and Limits

As we defined a dual notion to a universal arrow, it comes to no surprise that we can also define a dual notion to colimits and coproducts, namely limits and products.

Definition 4.1. Given categories C, J and the diagonal functor $\Delta : C \to C^J$, a limit for a functor $F : J \to C$ is a universal arrow $\langle r, v \rangle$ from ΔtoF. $r = limF$ is an object in C, called the limit object for the functor F, and $v : \Delta r \to F$ is a natural transformation universal among natural transformations $\tau : \Delta c \to F$.

τ consists of one arrow $\tau_i : c \to F_i$ for each $i \in J$ such that for every arrow $u : i \to j$, of J, one has $\tau_j = Fu \circ \tau_i$. When drawn, this is the cone from the vertex $LimF$ to the base F. That it is a universal can be seen in the diagram on page 68, and below:

$$\begin{tikzcd} c \arrow[r, "t"] \arrow[d, "\tau_i"] \arrow[dr, "\tau_j"] & LimF \arrow[d, "v_j"] \arrow[dl, "v_i"] \\ F_i \arrow[r, "F_u"] & F_j \end{tikzcd}$$

As before, limiting cones are unique up to isomorphism. The following diagram sums up how Colimits and Limits interact:

$$\begin{tikzcd} c \arrow[r, "\tau"] \arrow[d, dotted] & F \arrow[r, "\sigma"] & c \arrow[d, dotted] \\ LimF \arrow[r, "v"] & F \arrow[r, "\mu"] & ColimF \end{tikzcd}$$

Additionally, when the limits exist, we have natural isomorphisms

$$C(c, Limf) \cong Nat(\Delta c, F) = Cone(c, F)$$

$$Cone(F, c) = Nat(F, \Delta c) \cong C(ColimF, c)$$

Now we may talk about the duals of those special colimits.

Definition 4.2. If J is the discrete category $\{1, 2\}$, a functor $F : 1, 2 \to C$ is a pair of objects $\langle a, b \rangle$ of C. The limit object object is called a *product* of $aandb$, denoted by $a \times b$ or $a \Pi b$.

From each $a \times b$, there are projection arrows, denoted p, q such that $p : a \times b \to a$ and $q : a \times b \to b$. This constitutes a cone from the vertex $a \times b$ (its a very simple cone so don't strain yourself too much), and also gives us a bijection of sets

$$C(c, a \times b) \cong C(c, a) \times C(c, b)$$

natural in c, sending each $h : c \to a \times b$ to the pair $\langle ph, qh \rangle$. Conversely, when given arrows $f : c \to a$ and $g : c \to b$, there is a unique $h : c \to a \times b$ with $ph = f, qh = g$.

This should all be very familiar, and I don't think it is difficult to construct an example in **Set** with your imagination. I will skip *Infinite Products* and *Powers* because I don't think they are very interesting. If you want a quick overview of them, return to the corresponding Colimit subsections and flip all the arrows. These two behave exactly as you would expect as duals to the Colimit subsections.

Definition 4.3. If $J = \downarrow\downarrow$, a functor $F : \downarrow\downarrow \to C$ is a pair $f, g : b \to a$ of parallel arrows of C. If it exists, the limit object d of F is called an *equalizer* with diagram as follows: $d \xrightarrow{e} b \overset{f,g}{\rightrightarrows} a$ $(fe = ge)$

With $e : d \to b$ the limit arrow, called the equalizer of f and g, for any $h : c \to b$ with $fh = -gh$, there is a unique $h' : c \to d$ with $eh' = h$.

Consider the category **Set**. Then d is the set $\{x \in b | fx = gx\}$ with $e : d \to b$ the injection map. Even better, the equalizer always exists in **Set**, which is not necessarily true of all categories. Finally, let's talk about the best special case of a Limit:

Definition 4.4. If $J = (\to \bullet \leftarrow)$(the category I denoted as Q earlier), a functor $F : (\to \bullet \leftarrow) \to C$ is a pair of arrows $f : b \to a, g : a \to d$ of C with common codomain a. A commutative square is then of the form shown on the right:

$$\begin{tikzcd} c \arrow[r, "k"] \arrow[d, "h"] & d \arrow[d, "g"] \\ b \arrow[r, "f"] & a \end{tikzcd} \qquad \begin{tikzcd} b \times_a d \arrow[r, "q"] \arrow[d, "p"] & d \arrow[d, "g"] \\ b \arrow[r, "f"] & a \end{tikzcd}$$

Such that for any square with vertex c similar to the square on the left, there is a unique $r : c \to b \times_a d$ with $k = qr, h = pr$. This square formed by the universal cone is called the *pullback square* and the vertex $b \times_a d$ of the universal cone is called a *pullback*.

Similar to pushouts, the pullback of a pair of equal arrows $f : b \to a \leftarrow b : f$ is called the kernel pair of f. it works similar to cokernel pairs, and if you want to know more, I would suggest reading page 71.

Final notes include the limits on the empty category, i.e. $J = \mathbf{0}$. it shouldn't be much to convince you that the limit of the empty functor to C is a terminal object of C. In addition to this, one can also talk about limits of diagrams, but you essentially get the same information, so I would recommend sticking to the methods illustrated above. If, for some reason, you need this information though, the bottom of page 71 contains what Maclane says on the subject matter.