VANISHING OF L-FUNCTIONS AND RANKS OF SELMER GROUPS

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Abstract. This paper connects the vanishing at the central critical value of the L-functions of certain polarized regular motives with the positivity of the rank of the associated p-adic (Bloch-Kato) Selmer groups. For the motives studied it is shown that vanishing of the L-value implies positivity of the rank of the Selmer group. It is further shown that if the the order of vanishing is positive and even then the Selmer group has rank at least two. The proofs make extensive use of families of p-adic modular forms. Additionally, the proofs assume the existence of Galois representations associated to holomorphic eigenforms on unitary groups over an imaginary quadratic field.

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0. Introduction

This paper aims to connect the order of vanishing of the L-functions of certain (motivic p-adic) Galois representations with the ranks of their associated Selmer groups. This connection, really an assertion of equality, is part of the general Bloch-Kato conjectures (cf. [BK] and [FP]), but its orgins are in the 'class number formula' for number fields - part of which is the assertion that the order of vanishing at $s = 0$ of the Dedekind zeta-function $\zeta_K(s)$ of a number field K equals the rank of the group of units of K - and the celebrated conjecture of Birch and Swinnterton-Dyer - which asserts that the order of vanishing at $s = 1$ of the L-function $L(E, s)$ of an elliptic curve over a number field K equals the rank of the Mordell-Weil group $E(K)$. In both of these instances the equality can be restated in terms of ranks of Selmer groups (in the case of elliptic curves this requires finiteness of the (p-primary part of the) Tate-Shafarevich group of the curve).

In this paper we work in the context of a polarized regular (pure motivic) Galois representation $R : G_{\mathcal{K}} \to \mathrm{GL}_d(L)$ of the absolute Galois group $G_{\mathcal{K}}$ of an imaginary quadratic field $\mathcal{K}$ defined over a p-adic field L; we fix a prime p that splits in $\mathcal{K}$. The polarization condition is an isomorphism

$$R^\vee(1) \cong R^c$$

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of the arithmetic dual of R with the conjugate of R by the non-trivial automorphism c of $\mathcal{K}$. The (motivic and) regular condition is that R is Hodge-Tate at the primes above p and that the Hodge-Tate weights are regular. We further restrict to the case where the Hodge-Tate weights of R do not include 0 or -1. Unfortunately, this excludes the case of elliptic curves. The L-functions $L(R,s)$ of such Galois representations, defined using geometric Frobenius elements (throughout we adopt geometric conventions), are expected to have meromorphic continuations to all of $\mathbf{C}$ and to satisfy the functional equation

$$L(R,s) = \epsilon(R,s)L(R^\vee, 1-s).$$

The value $s=0$ is a critical value of $L(R,s)$, and the connection between orders of vanishing and ranks of Selmer groups is the following.

Conjecture. $\mathrm{ord}_{s=0}L(R,s) = \mathrm{rank}_L H^1_f(\mathcal{K}, R^\vee(1))$.

Here $H^1_f(\mathcal{K}, R^\vee(1)) \subseteq H^1(\mathcal{K}, R^\vee(1))$ is the Bloch-Kato Selmer group. This is defined by imposing local conditions at all primes. At primes not dividing p the classes are required to be unramified, while at primes v dividing p they are required to be crystalline: their image in $H^1(I_v, R^\vee(1)\otimes B_{cris})$ is zero, where B_{cris} is Fontaine's ring of p-adic periods.

The Galois representations we consider are expected to be automorphic in the sense that for a given R there should exist a unitary group $U(V)$ in d-variables, an automorphic representation π of $U(V)$ with infinity-type a holomorphic discrete series, and an algebraic idele class character χ of $\mathcal{K}$ satisfying $\chi|^\times_{\mathbf{A_Q}} = |\cdot|^{2\kappa'}_{\mathbf{A_Q}}$ such that $L(R,s) = L(\pi, \chi^{-1}, s+\kappa'+1/2)$, where the right-hand side is a twist of the standard L-function of π. Such an identification is generally the only known strategy for proving the conjectured analytic properties of $L(R,s)$. So we start by assuming that given π and χ, the corresponding R exists. In general this is only known for unitary groups in 3 or fewer variables (see [BR92]) or under certain local hypotheses on π (see [HL04]) (these conditions certainly do not hold in all the cases we consider). We further assume that π and χ are unramified at primes above p. We then prove two theorems - Theorems 4.3.1 and Theorems 5.1.1 - in the direction of the above conjecture. We emphasize that their proofs require the existence of Galois representations associated to certain cuspidal representations of unitary groups; this existence is made precise in Conjecture 4.1.1. The first of these theorems is the following.

Theorem A. *If* $L(\pi, \chi^{-1}, \kappa'+1/2) = L(R,0) = 0$ *then* $\mathrm{rank}\, H^1_f(\mathcal{K}, R^\vee(1)) \geq 1$.

We include a few remarks about this theorem.

(i) In earlier work [SU02], [SU06] we proved a result similar to Theorem A: if F is a holomorphic modular form of even weight $2k$ and trivial nebentypus and ordinary for p and if $\mathrm{ord}_{s=k}L(F,s)$ is odd then the rank of the corresponding p-adic Selmer group $H^1_f(\mathbf{Q}, V_F(k))$ is positive (V_F is the p-adic Galois representation associated to F). The positivity of the rank in the case of even order vanishing - at least if F is unramified at p - will follow from our forth-coming work [SU-MC] on the Iwasawa main conjecture for modular forms[1]. For prior results in the same vein (by Gross and Zagier, Greenberg, Nekovář, Bellaïche...) the interested reader should consult the introduction to [SU06].

(ii) When π is just an idele class character, so R is one-dimensional, Theorem A is unconditional. Since no hypothosis is imposed on the epsilon factor $\epsilon(R,0)$, Theorem A in this case generalizes the complex multiplication case of [SU02], [SU06], where $\epsilon(R,0) = -1$ is required (the $\epsilon(R,0) = -1$ case is also the main result of [BC04]; our proof of Theorem A provides an alternate proof of this case).

(iii) Suppose F is a holomorphic modular form of even weight $2k > 2$, trivial nebentypus, and level prime to p. One consequence of Theorem A is that if $L(F,k) = 0$, then the rank of $H^1_f(\mathcal{K}, V_F(k))$ is positive. Choosing $\mathcal{K}$ so that the twist $F_{\mathcal{K}}$ of F by the character of $\mathcal{K}$ is such that $L(F_{\mathcal{K}}, k) \neq 0$ and appealing to a result of Kato [Ka04] that asserts $H^1_f(\mathbf{Q}, V_{F_{\mathcal{K}}}(k)) = 0$ in this case, we can then conclude that $H^1_f(\mathbf{Q}, V_F(k))$ has positive rank. This provides another proof of the results from remark (i) as well as an extension of them to the non-ordinary case.

(iv) The authors of [BC04] have announced a result in the spirit of Theorem A but with a number of additional hypotheses, including $\epsilon(R,0) = -1$, and certain of the Arthur conjectures.

Our proof of the Theorem A follows along the same lines as the proof of the main result in [SU02], [SU06]. As explained in §1, the vanishing of the L-function at $s = 0$ implies the existence of a holomorphic Eisenstein series on a larger unitary group. This is analogous to the situation in *loc. cit.* where odd-order vanishing implies the existence of a special cuspform on a larger group, there a symplectic group of genus 2. In §§2 and 3 we construct a p-adic deformation of this Eisenstein series, a p-adic family of automorphic representations containing the Eisenstein representation. The generic member of this family is cuspidal. The Galois representations associated to these cuspidal representations (whose existence is one of our primary hypotheses) are generically irreducible. Putting all this together, we construct an irreducible family of Galois representations that specializes at one point to the reducible Galois representation $1 \oplus \epsilon_p \oplus R$ (the Galois representation of the Eisenstein series). By a now standard argument, we then deduce the existence of a non-trivial $G_{\mathcal{K}}$-extension $0 \to L(1) \to E \to R \to 0$. Using a result of Kisin [Ki] we are able to deduce that this extension lies in $H^1_f(\mathcal{K}, R^\vee(1))$.

In the last section of this paper we extend Theorem A to a higher-rank case (under the same hypotheses on χ and π).

Theorem B. *If* $\mathrm{ord}_{s=0} L(\pi, \chi^{-1}, s + \kappa' + 1/2) = \mathrm{ord}_{s=0} L(R, s)$ *is even and positive, then* $\mathrm{rank}\, H^1(\mathcal{K}, R^\vee(1)) \geq 2$*.*

The proof of Theorem B relies on that of Theorem A. The hypothesis that $L(R,s)$ vanishes to even order at $s = 0$ means that $\epsilon(R,0) = 1$. And so the epsilon factor of the

[1]This work includes a local hypothesis on the modular form F and its associated mod p Galois representation.

primitive L-function of the Eisenstein series constucted in the proof of Theorem A is equal to -1. This is then true of all the cuspidal representations in the p-adic family from the proof of that theorem. In particular, their L-functions satisfy the hypothesis of Theorem A. So running through the proof of that theorem for these cuspidal representations, one deduces the existence of a p-adic family of generically irreducible Galois representations which specializes at one point to the representation $L^2 \oplus L(1)^2 \oplus R$. And then from this we deduce the existence of a subspace of rank 2 in the Selmer group.

The proofs of Theorems A and B rely crucially on the theory of p-adic families of automorphic representations, especially as developed in [KL] and in [U06].

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Standard notation. Throughout this paper p is a fixed prime. Let $\overline{\mathbf{Q}}$ and $\overline{\mathbf{Q}}_p$ be, respectively, algebraic closures of $\mathbf{Q}$ and $\mathbf{Q}_p$ and let $\mathbf{C}$ be the field of complex numbers. We fix embeddings $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Throughout we implicitly view $\overline{\mathbf{Q}}$ as a subfield of $\mathbf{C}$ and $\overline{\mathbf{Q}}_p$ via the embeddings ι_∞ and ι_p. Let $\mathbf{C}_p$ be the completion of $\overline{\mathbf{Q}}_p$ with respect to its p-adic metric. We fix an identification $\mathbf{C}_p \cong \mathbf{C}$ compatible with the embeddings ι_p and ι_∞.

We fix $\mathcal{K} \subset \overline{\mathbf{Q}}$ an imaginary quadratic field. We denote by c the complex conjugation of $\mathbf{C}$ (and hence of $\mathcal{K}$). We assume that p splits in $\mathcal{K}$: $p = \wp\wp^c$ with $\wp$ the prime ideal of $\mathcal{K}$ induced by ι_p. We write ϖ for an uniformizer of $\wp$.

1. Eisenstein series and vanishing of L-functions

1.1. **Unitary groups.** Let θ be a totally imaginary element in $\mathcal{K}$ such that $-i\theta > 0$ and let $\Delta = \theta\bar{\theta}$ (a positive rational number). In sections §§2-5 we will assume that $\mathrm{ord}_p(\Delta) = 0$. Given integers $b \geq a \geq 0$, $a + b = d > 0$, we let

$$T_{a,b} = \begin{pmatrix} & & 1_b \\ & \theta^{-1} & \\ -1_b & & \end{pmatrix} \in \mathrm{GL}_d(\mathcal{K}).$$

We let $G_{a,b}$ be the unitary group associated to this (skew-Hermitian) matrix: for any $\mathbf{Q}$-algebra R

$$G_{a,b}(R) = \{g \in \mathrm{GL}_d(\mathcal{K} \otimes R) \ : gT_{a,b}{}^t\bar{g} = T_{a,b}\}.$$

Then $G_{a,b}(\mathbf{R})$ is a real unitary group of signature (a, b). The unbounded symmetric domain associated to this group is

$$\mathcal{D}_{a,b} = \{\begin{bmatrix} z \\ u \\ 1_a \end{bmatrix} \in M_{d\times a}(\mathbf{C}) \ : \ z \in M_{a\times a}(\mathbf{C}), u \in M_{(b-a)\times a}(\mathbf{C}), \theta^{-1}(z - z^*) - u^*u > 0\}.$$

The action of $G_{a,b}(\mathbf{R})$ on $\mathcal{D}_{a,b}$ is defined as follows: for $g \in G_{a,b}(\mathbf{R})$ and $x \in \mathcal{D}_{a,b}$

$$g(x) = g \cdot x \cdot t^{-1}, \quad g \cdot x = \begin{bmatrix} r \\ s \\ t \end{bmatrix}, \ r, t \in M_{a\times a}(\mathbf{C}),$$

where $\cdot$ denotes the usual matrix multiplication. Let

$$x_0 = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \in \mathcal{D}_{a,b}.$$

The stabilizer of x_0 in $G_{a,b}(\mathbf{R})$ is a maximal compact, which we denote $K_{a,b}$. This is the group of $\mathbf{R}$-points of an $\mathbf{R}$-group that we also denote by $K_{a,b}$. The map $g \mapsto g(x_0)$ is a real analytic isomorphism of $G_{a,b}(\mathbf{R})/K_{a,b}$ with $\mathcal{D}_{a,b}$. We will often write an element g of $G_{a,b}$ or $M_{d\times d}$ in block form: $g = (g_{ij})_{1\leq i,j\geq 3}$ with $g_{11}, g_{33} \in M_{a\times a}$. We let $B_{a,b}$ be the $\mathbf{Q}$-rational Borel of $G_{a,b}$ defined by requiring $g_{21} = g_{31} = g_{32} = 0$ and g_{33} to be upper-triangular (so g_{11} is lower-triangular).

Let

$$c = c_{a,b} = 2^{-1/2}\begin{pmatrix} 1_a & & -i1_a \\ & \sqrt{|\theta|}1_{b-a} & \\ -i1_a & & 1_a \end{pmatrix} \in \mathrm{GL}_d(\mathcal{K}).$$

Then $cT_{a,b}{}^t\bar{c} = i/2\mathrm{diag}(1_a, -1_b)$, so $k \mapsto ckc^{-1}$ identifies $K_{a,b}$ with the $\mathbf{R}$-group $U(a) \times U(b)$ (embedded diagonally in $\mathrm{GL}_d(\mathcal{K})$). Let $H_{a,b}$ be the Cartan subgroup of $K_{a,b}$ that is identified with the group of diagonal matrices in $U(a)\times U(b)$. Let $J_{a,b} : G_{a,b}(\mathbf{R})\times\mathcal{D}_{a,b} \to K_{a,b}(\mathbf{C})$ be the canonical automorphy factor: if $k \in K_{a,b}$ then $J_{a,b}(k, x_0) = k$, and

$$cJ_{a,b}(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{pmatrix}, x_0)c^{-1} = (a_{11}, \begin{pmatrix} a_{22} & a_{23} \\ & a_{33} \end{pmatrix}).$$

These properties, together with the usual cocycle condition, completely determine $J_{a,b}$.

We also fix for each prime ℓ a maximal compact $K_{a,b,\ell} \subset G_{a,b}(\mathbf{Q}_\ell)$, and let $K_{a,b,f} = \prod K_{a,b,\ell}$.

Let $a' = a + 1, b' = b + 1$. Let $P_{a,b}$ be stabilizer in $G_{a',b'}$ of the line $\{(0, ..., 0, x) \in \mathcal{K}^{d+2} : x \in \mathcal{K}\}$. Then $P_{a,b}$ is a standard, maximal $\mathbf{Q}$-parabolic of $G_{a',b'}$ with standard Levi subgroup $L_{a,b}$ isomorphic to $G_{a,b} \times Res_{\mathcal{K}/\mathbf{Q}}\mathbf{G}_m$: a pair $(g, t) \in G_{a,b} \times Res_{\mathcal{K}/\mathbf{Q}}\mathbf{G}_m$ is identified with

$$m(g, t) = \begin{pmatrix} g_{11} & & g_{12} & g_{13} & \\ & \bar{t}^{-1} & & & \\ g_{21} & & g_{22} & g_{23} & \\ g_{31} & & g_{32} & g_{33} & \\ & & & & t \end{pmatrix} \in G_{a',b'}.$$

We write $N_{a,b}$ for the unipotent radical of $P_{a,b}$. The map $r_{a,b} : \mathcal{D}_{a',b'} \to \mathcal{D}_{a,b}$ given by

$$r_{a,b}(\begin{bmatrix} z \\ u \\ 1_{a'} \end{bmatrix}) = \begin{bmatrix} z' \\ u' \\ 1_a \end{bmatrix},$$

$$z = (z_{ij})_{1\leq i,j\leq a+1}, \quad z' = (z_{ij})_{1\leq i,j\leq a},$$
$$u = (u_{i,j}), \quad u' = (u_{i,j})_{j\leq a},$$

is $P_{a,b}(\mathbf{R})$-equivariant in the sense that if $p = m(g, t)n \in P_{a,b}(\mathbf{R}) = L_{a,b}(\mathbf{R})N_{a,b}(\mathbf{R})$ then $r_{a,b}(p(x)) = g(r_{a,b}(x))$.

The algebraic characters of $G_{a,b}$ correspond to d-tuples of integers $(c_d, ..., c_{b+1}; c_1, ..., c_b)$ in the usual way. The irreducible algebraic representations of $K_{a,b}$ are then classified by those d-tuples satisfying $c_1 \geq c_2 \geq \cdots \geq c_b$ and $c_{b+1} \geq c_{b+2} \geq c_d$. Such d-tuples also classify the L-packets of discrete series representations of $G_{a,b}(\mathbf{R})$. The holomorphic discrete series correspond to those d-tuples such that $c_b - c_{b+1} \geq d$. Given such a d-tuple τ, we write π^H_τ for the correspoding holomorphic discrete series.

When a and b are fixed or their exact values unimportant, we write G for $G_{a,b}$ and H for $G_{a',b'}$. In our remaining notation we drop the subscript 'a, b' and replace the subscript 'a', b'' with with a superscript $'$.

1.2. **L-functions.** Let π be an automorphic representation of G and χ an idele class character of $\mathbf{A}_{\mathcal{K}}^\times$. We write $L(\pi,\chi,s)$ for the standard L-function associated to π and χ: if $BC(\pi)$ is the formal base change of π to $\mathrm{GL}(d)_{/\mathcal{K}}$ then $L(\pi,\chi,s) = L(BC(\pi),\chi,s)$. If S is a finite set of places of $\mathbf{Q}$ then the superscript 'S' on $L^S(\pi,\chi,s)$ will, as usual, mean that the Euler factors at the places in S have been omitted. If $d>1$ and π is cuspidal and not endoscopic or CAP, then $BC(\pi)$ is expected to be cuspidal, hence the L-functions $L^S(\pi,\chi,s)$ are expected to satisfy the following:

$$L^S(\pi,\chi,s) \text{ is holomorphic on all of } \mathbf{C} \tag{1.2.1}$$

and

$$\text{if } \pi \text{ and } \chi \text{ are unitary, then } L^S(\pi,\chi,s)\neq 0 \text{ for } \mathrm{Re}(s)\geq 1. \tag{1.2.2}$$

Remark 1.2.1. That $BC(\pi)$ is cuspidal as expected is known in certain cases: (1) if $d=2,3$ or (2) if π_v is supercuspidal for some finite place v.

1.3. **Eisenstein series.** Given a cuspidal automorphic representation π of G with underlying space V_π and an idele class character χ of $\mathbf{A}_{\mathcal{K}}^\times$, we let $\rho=\rho_{\pi,\chi}$ be the representation of $P(\mathbf{A})$ on V_π defined by $\rho(m(g,t)n)v = \chi(t)\pi(g)v$, $m(g,t)\in M(\mathbf{A})$, $n\in N(\mathbf{A})$. Let $I(\rho)$ be the space of smooth, K'-finite functions $f: H(\mathbf{A})\to V_\pi^{sm}$ such that $f(pg)=\rho(p)f(g)$. We assume that V_π has been identified with a cuspidal subspace of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$, so the smooth vectors V_π^{sm} are smooth functions and the smooth, K-finite vectors $V_\pi^{sm,fin}$ are cuspforms. Then evaluation at the identity converts $f\in I(\rho)$ into a $\mathbf{C}$-valued function on $H(\mathbf{A})$; we often write $f(x)$ for $f(x)(1)$. Bearing this in mind, given $f\in I(\rho)$ and a complex number s we consider the Eisenstein series

$$E(f;s,g) = \sum_{\gamma\in P(\mathbf{Q})\backslash H(\mathbf{Q})} f(\gamma g)\delta(\gamma g)^{s+1/2},$$

where δ is the usual modulus function for P: $\delta(m(g,t)) = |t\bar{t}|_{\mathbf{A}}^{-(d+1)}$. If $\mathrm{Re}(s)$ is sufficiently large (if π and χ are unitary and π is tempered then $\mathrm{Re}(s)>1/2$ suffices) then this series converges absolutely and uniformly for s and g in compact sets and so is holomorphic in s and defines an automorphic form on $H(\mathbf{A})$. The general theory of Eisenstein series provides a meromorphic continuation of $E(f;s,g)$ to all of $\mathbf{C}$.

1.4. **Holomorphy and vanishing of L-functions.** Suppose that $\pi=\otimes\pi_v$ is such that $\pi_\infty = \pi_\tau^H$ for some d-tuple $\tau=(c_d,\ldots,c_{b+1};c_1,\ldots,c_b)$. We identify τ with the corresponding algebraic representation of K and write V_τ for the complex points of the underlying module (so V_τ is a finite-dimensional complex vector space and τ defines an action of $K(\mathbf{C})$ on V_τ). Then $(V_{\pi_\infty}^{sm,fin}\otimes V_\tau)^K$ is one-dimensional. Let φ_∞ be a non-zero generator of this space. Let $\varphi_f\in\otimes_{\ell\neq\infty}V_{\pi_\ell}^{sm}$ and let $\varphi=\varphi_\infty\otimes\varphi_f\in V_\pi^{sm,fin}\otimes V_\tau$. We convert φ into something more classical as follows. We write $\tau(g,x)$ for $\tau(J(g,x))$ and set

$$F(Z) = \tau(g,x_0)\varphi(gx),\quad g\in G(\mathbf{R}), g(x_0)=Z\in\mathcal{D}_G.$$

This is a holomorphic function of Z, and if $U \subseteq G(\mathbf{A}_f)$ is an open compact such that $\varphi(gk) = \varphi(g)$ for all $k \in U$ then F satisfies

$$F(\gamma(Z)) = \tau(\gamma, Z)F(Z), \quad \gamma \in \Gamma = G(\mathbf{Q}) \cap U.$$

Let $\chi = \otimes\chi_v$ be an idele class character of $\mathbf{A}_{\mathcal{K}}^\times$ such that $\chi_\infty = z^n\bar{z}^m$ with $n+m$ having the same parity as d. Let $\kappa, \kappa' \in \frac{1}{2}\mathbf{Z}$ be defined by $2\kappa = n - m$ and $2\kappa' = n + m$ and assume κ satisfies

$$c_b \geq \kappa + d/2 + 1, \quad \kappa - d/2 - 1 \geq c_{b+1}. \tag{1.4.1}$$

Let ξ be the $d+2$-tuple $\xi = (c_d, ..., c_{b+1}, \kappa - d/2 - 1; c_1, ..., c_b, \kappa + d/2 + 1)$. As in the case of τ, we identify ξ with the corresponding algebraic representation of K' and write V_ξ for the complex points of the underlying module. The representation τ appears with multiplicity one in the restriction of ξ to K, the latter viewed as a subgroup of K' via $k \mapsto m(k, 1)$; the other irreducible representations appearing in this restriction have highest weight dominated by τ. We fix a K-equivariant inclusion of V_τ into V_ξ (explicitly, if v and w are respective highest weight vectors of these representations then $v \mapsto w$ determines such an inclusion). Then $(V_\pi^{sm,fin} \otimes V_\xi)^K = (V_\pi^{sm,fin} \otimes V_\tau)^K$ since τ is the minimal K-type in π_∞.

There are compatible factorizations $\rho = \otimes\rho_v$ and $I(\rho) = \otimes I(\rho_v)$, with $\rho_v = \rho_{\pi_v,\chi_v}$ and $I(\rho_v)$ defined similary to ρ and $I(\rho)$. Let $\rho_f = \otimes_{v\neq\infty}\rho_v$ and $I(\rho_f) = \otimes_{v\neq\infty}I(\rho_v)$. A straight-forward application of Frobenius reciprocity shows that $(I(\rho_\infty) \otimes V_\xi)^{K'}$ is one-dimensional. Let Φ_∞ be a generator of this space. Let $\Phi_f \in I(\rho_f)$ and let $\Phi = \Phi_\infty \otimes \Phi_f \in (I(\rho) \otimes V_\xi)^{K_{H,\infty}}$. For $h \in H(\mathbf{A}_f)$, $\Phi(h) \in (V_\pi^{sm,fin} \otimes V_\xi)^{K_{G,\infty}} = (V_\pi^{sm,fin} \otimes V_\tau)^{K_{G,\infty}}$. Let $\varphi_h = \Phi(h)$. Then $\varphi_h = \varphi_\infty \otimes \varphi_{h,f}$.

We relate Φ to something more classical as we did φ (and hence each φ_h). For $g \in H(\mathbf{R})$ and $Z \in \mathcal{D}'$ we let $\xi(g, Z) = \xi(J'(g, Z))$. For $h \in H(\mathbf{A}_f)$ and $s \in \mathbf{C}$ we then set

$$\mathcal{F}_h(s, Z) = \xi(g, x_0)\Phi(gh)\delta(g)^{s+1/2}, \quad g \in H(\mathbf{R}), g(x_0) = Z \in \mathcal{D}'.$$

If $U \subseteq H(\mathbf{A}_f)$ is an open compact such that $\Phi_f(gkh) = \Phi_f(gh)$ for all $k \in U$, then $\mathcal{F}_h$ satisfies

$$\mathcal{F}_h(s, p(Z)) = \xi(p, Z)\mathcal{F}_h(s, Z), \quad p \in P(\mathbf{Q}) \cap U.$$

It follows from the definition of $\mathcal{F}_h(s, Z)$ that if $p = m(g, t)n \in P(\mathbf{R})$ is such that $p(x_0) = Z$ (such a p always exists since $H(\mathbf{R}) = P(\mathbf{R})K_H$), then

$$\mathcal{F}_h(s, Z) = (t\bar{t})^{1/2+\kappa'-s(d+1)}F_h(r(Z)),$$

where F_h is the function on $\mathcal{D}$ associated to $\varphi_h = \Phi(h) \in (V_\pi^{sm,fin} \otimes V_\xi)^K$ as above. In particular, if

$$s_0 = (1/2 + \kappa')/(d + 1)$$

then $\mathcal{F}_h(s_0)$ is visibly holomorphic as a function on $\mathcal{D}'$.

Let $v_1, ..., v_n$ be a basis for V_ξ and write $\Phi_\infty = \sum \Phi_{\infty,i} \otimes v_i$ with $\Phi_{\infty,i} \in I(\rho_\infty)$. Put $\Phi_i = \Phi_{\infty,i} \otimes \Phi_f$ and $E(\Phi; s, g) = \sum E(\Phi_i; s, g) \otimes v_i$ and

$$E(\mathcal{F}_h; s, Z) = \xi(g, x_0)E(\Phi; s, gh), \quad g \in H(\mathbf{R}), g(x_0) = Z.$$

This last is a V_ξ-valued Eisenstein series on $\mathcal{D}'$. It is holomorphic at $s \in \mathbf{C}$ if $E(\Phi; s, g)$ (so if each $E(\Phi_i; s, g)$ is).

We denote by $\eta_\mathcal{K}$ the quadratic Dirichlet character attached to the extension $\mathcal{K}/\mathbf{Q}$. Then we have the following proposition.

Proposition 1.4.1. *Suppose π is the twist of a tempered representation and suppose* (1.2.1) *and* (1.2.2) *hold.*

(i) *The series $E(\mathcal{F}_h; s, Z)$ is holomorphic as a function of s at $s = s_0$.*
(ii) *If $\chi|_{\mathbf{A}_\mathbf{Q}^\times} \neq |\cdot|_\mathbf{A}^{2\kappa'}\eta_\mathcal{K}^d$ or if $L(\pi, \chi^{-1}, 1/2 + \kappa') = 0$ then $E(\mathcal{F}_h; Z) = E(\mathcal{F}_h; s_0, Z)$ is holomorphic as a function of Z.*

Remark 1.4.2. An important observation is that no hypotheses have been imposed on the section Φ_f. In practice we will assume π and χ to be unramified at p and take the p-component of Φ_f to be a certain 'p-stabilization' of the spherical vector, chosen to be amenable to methods of p-adic deformations of modular forms. At the primes different from p we will generally take Φ_f to be as unramified as possible.

Proof. We briefly indicate a proof of Proposition 1.4.1. Parts (i) and (ii) both follow from analyzing the constant terms of the Eisenstein series $E(\Phi_i; s, g)$ and $E(\Phi; s, g)$. First we note that since π is cuspidal and P is maximal, the constant term along a standard parabolic other than P or G is zero. The constant term of $E(\Phi_i; s, g)$ along P can be expressed in terms of the image of Φ_i under a certain intertwining operator, as we now recall.

Implicit in the factorization $V_\pi = \otimes V_{\pi_v}$ is the choice of a newvector $\phi_v \in V_{\pi_v}^{K_v}$ at each v at which π_v is unramified. If π_v and χ_v are both unramified then we let $\Phi_v^{sph} \in I(\rho_v)^{K'_v}$ be the generator such that $\Phi_v^{sph}(1) = \phi_v$. The factorization $I(\rho) = \otimes I(\rho_v)$ is with respect to the Φ_v^{sph}'s.

Let $\rho^\vee$ and $I(\rho^\vee)$ be defined as ρ and $I(\rho)$ were but with χ replaced by $\chi^\vee = (\chi^c)^{-1}$), and let $\rho_v^\vee$ and $I(\rho_v^\vee)$, v a place of $\mathbf{Q}$, be similarly defined. If π_v and χ_v (and hence $\chi_v^\vee$) are unramified at v, the we let $\Phi_v^{\vee, sph} \in I(\rho_v^\vee)^{K'_v}$ be such that $\Phi_v^{\vee, sph}(1) = \phi_v$. We let $\Phi_\infty^\vee \in (I(\rho_\infty^\vee) \otimes V_\xi)^{K'}$ be a non-zero generator and write $\Phi_\infty^\vee = \sum \Phi_{\infty,i}^\vee \otimes v_i$.

For $\phi \in I(\rho)$ or $I(\rho^\vee)$ and $s \in \mathbf{C}$ we let $\phi_s = \phi\delta^{s+1/2}$. The constant term of $E(\Phi_i; s, g)$ along P is $\Phi_{i,s} + M(s, \Phi_i)_{-s}$ where $M(s, -) : I(\rho) \to I(\rho^\vee)$ is the usual intertwining operator associated to P; this is meromorphic as a function of s and for $\mathrm{Re}(s)$ sufficiently large it is defined by the integral

$$M(s, \varphi)_{-s}(g) = \int_{N(\mathbf{A})} \varphi_s(wng)dn, \quad w = \begin{pmatrix} 1_a & & \\ & & 1 \\ & 1_b & \\ -1 & & \end{pmatrix} \in H(\mathbf{Q}). \tag{1.4.1}$$

We let $M_v(s, -) : I(\rho_v) \to I(\rho_v^\vee)$ be the usual local intertwining operator associated to P; for $\mathrm{Re}(s)$ sufficiently large, but independent of v, these are given by the local versions of the integral (1.4.1). If $\varphi = \otimes\varphi_v$ we then have $M(s, \varphi) = \otimes M_v(s, \varphi_v)$, provided the right-hand side converges.

Let $\eta_{\mathcal{K}_v}$ the quadratic character of $\mathbf{Q}_v^\times$ attached to the extension $\mathcal{K}_v/\mathbf{Q}_v$ if v is inert in $\mathcal{K}/\mathbf{Q}$ and the trivial character otherwise. For us, the important properties of the $M_v(s,-)$'s are

(a) if π_v and χ_v are unramified then

$$M_v(s,\Phi_v^{sph}) = \frac{L(\pi_v,\chi_v^{-1},(d+1)s)L(\chi_v'\eta_{\mathcal{K}_v}^d,(2d+2)s)}{L(\pi_c,\chi_v,(d+1)s+1)L(\chi_v'\eta_{\mathcal{K}_v}^d,(2d+2)s+1)}\Phi_v^{\vee,sph},$$

where $\chi_v' = \chi_v^{-1}|_{\mathbf{Q}_v^\times}$;

(b) for a finite place v, $M_v(s,-)$ is holomorphic at $s=s_0$;

(c) $\sum M_\infty(s,\Phi_{\infty,i})\otimes v_i = c(s)\Phi_\infty^\vee$, where $c(s)$ is a meromorphic function with a simple zero at $s=s_0$.

Part (a), of course, is a well-known computation. Part (b) follows from [Sh] and the hypothesis that π is a twist of a tempered representation. Part (c) is a relatively straightforward computation.

Suppose $\Phi_f = \otimes\Phi_\ell$; we may assume this without loss of generality since any Φ_f is a linear combination of such. Let S be the set of primes ℓ such that π_ℓ or χ_ℓ is ramified or $\Phi_\ell \neq \Phi_\ell^{sph}$. From (a) and (c) above it follows that for $\mathrm{Re}(s)$ sufficiently large we then have

$$M(s,\Phi_i) = \frac{c(s)L^S(\pi,\chi^{-1},(d+1)s)L^S(\chi'\eta_{\mathcal{K}}^d,(2d+2)s)}{L^S(\pi,\chi^{-1},(d+1)s+1)L^S(\chi'\eta_{\mathcal{K}}^d,(2d+2)s+1)}\Phi_{\infty,i}^\vee\otimes_{\ell\notin S}\Phi_\ell^{sph}\otimes_{\ell\in S}M_\ell(s,\Phi_\ell).$$

Note that $\chi' = \chi^{-1}|_{\mathbf{A}_\mathbf{Q}^\times}$ is an idele class character of $\mathbf{A}_\mathbf{Q}^\times$ with infinity type $z^{-2\kappa'}$. Thus $L^S(\chi'\eta_{\mathcal{K}}^d,(2d+2)s)$ is holomorphic at $s=s_0$ unless $\chi' = |\cdot|_\mathbf{A}^{-2\kappa'}\eta_{\mathcal{K}}^d$ in which case the L-function has a simple pole at $s=s_0$. It also follows that $L^S(\chi',(2d+2)s+1)$ is holomophic and non-zero at $s=s_0$. In particular, $c(s)L^S(\chi'\eta_{\mathcal{K}}^d,(2d+2)s)/L^S(\chi'\eta_{\mathcal{K}}^d,(2d+2)s+1)$ is holomorphic at $s=s_0$ and non-zero only if $\chi' = |\cdot|_\mathbf{A}^{-2\kappa'}\eta_{\mathcal{K}}^d$. It follows from (1.2.1) and (1.2.2) that $L^S(\pi,\chi^{-1},(d+1)s/L^S(\pi,\chi^{-1},(d+1)s+1)$ is holomorphic at $s=s_0$ and zero if and only if $L^S(\pi,\chi^{-1},1/2+\kappa')=0$. Putting all this together with (b) above we find that

(d) $M(s,\Phi_i)$ is holomorphic at $s=s_0$;

(e) $M(s,\Phi_i)=0$ if $\chi'\neq|\cdot|_\mathbf{A}^{-2\kappa'}$ or if $L(\pi,\chi^{-1},1/2+\kappa')=0$.

The general theory of Eisenstein series implies that $E(\Phi_i;s,g)$ is holomorphic at $s=s_0$ if the constant terms are. Thus (d) above implies the holomorphy of $E(\Phi_i;s,g)$, and hence of each $E(\mathcal{F}_h;s,Z)$, at $s=s_0$, proving part (i) of the proposition. It also follows from the general theory of Eisenstein series that $E(\mathcal{F}_h;s_0,Z)$ is holomorphic as a function of Z if its constant terms are. This is equivalent to the holomorphy of the functions

$$Z\mapsto \xi(g,x_0)\left(\Phi_s(gx)+M(s,\Phi)_{-s}(gx)\right),\quad g\in H(\mathbf{R}), g(x_0)=Z, x\in H(\mathbf{A}_f),$$

where $M(s,\Phi)=\sum M(s,\Phi_i)\otimes v_i$. If $\chi'\neq|\cdot|_{\mathbf{A}}^{-2\kappa'}\eta_{\mathcal{K}}^d$ or $L(\pi,\chi^{-1},1/2+\kappa')=0$, it follows from (e) above that this function equals $\mathcal{F}_a(s_0,Z)$ at $s=s_0$ and so is holomorphic. This proves part (ii) of the proposition. ■

2. p-adic deformations of automorphic representations

It is impossible to list here all the contributors to this area. However, we want to emphasize that the important recent developments grew from the seminal ideas of Hida, Coleman, Mazur and Stevens. For our application, we rely mostly on an approach that has been stressed in [U06]: instead of constructing a space interpolating spaces of automorphic forms, one directly studies the p-adic properties of the 'trace' distribution. This approach is analogous to Wiles' introduction of pseudo-representations for the study of deformations of Galois representations.

2.1. **Hecke operators.** In this paper we take a Hecke operator to be a compactly supported smooth $\mathbf{Q}$-valued function on $G(\mathbf{A}_f)$. We fix a Haar measure on $G(\mathbf{A}_f)$ such that the maximal compact K_f has volume 1. If (π,V_π) is a smooth representation, then the action of a Hecke operator on V_π is defined using this Haar measure.

We need to restrict attention to Hecke operators of specific types at the prime p. To describe these we first fix an isomorphism $G(\mathbf{Q}_p)\cong GL_d(\mathcal{K}_\wp)=GL_d(\mathbf{Q}_p)$ so that $g=(g_{ij})\in G(\mathbf{Q}_p)$ is identified with $g'=(g'_{ij})\in \mathrm{GL}_d(\mathcal{K}_\wp)$ with $g'_{11}={}^tg_{11}$ and $g'_{33}=g_{33}$ and so that $B(\mathbf{Q}_p)$ is identified with a standard parabolic of $\mathrm{GL}_d(\mathbf{Q}_p)$ (i.e., contains the subgroup of upper-triangular matrices). We assume that the maximal compact $K_p\subset G(\mathbf{Q}_p)$ is identified with $\mathrm{GL}_d(\mathbf{Z}_p)$.

For each positive integer m we let $I_m\subset GL_d(\mathbf{Z}_p)$ be the subgroup of matrices that are upper-triangular modulo p^m. Let $t=(t_1,\ldots,t_d)$ be a decreasing sequence of n integers. We denote by u_t the characteristic function on $GL_d(\mathbf{Q}_p)$ of the double class $I_m.\mathrm{diag}(p^{t_1},\ldots,p^{t_n}).I_m$. The u_t's generate a commutative algebra[2] via the convolution product. We denote this algebra by $\mathcal{U}_p$.

Let S be a set of finite primes containing p and the primes at which G is ramified. We let $K^S=\prod_{\ell\notin S}K_\ell\subset G(\mathbf{A}_f^S)$, a maximal compact open subgroup, and we put

$$\mathcal{R}_{S,p}:=\mathcal{C}_c^\infty(K^S\backslash G(\mathbf{A}_f^S)/K^S),\mathbf{Z})\otimes\mathcal{U}_p.$$

This Hecke operator acts naturally on any $V_\pi^{I_n.K^S}$.

2.2. **p-stabilizations and normalizations.** Let (π,V_π) be an automorphic representation such that $V_\pi^{K^S.I_n}\neq 0$ and $\pi_\infty\cong\pi_\tau^H$ with τ a d-tuple as in §1.1. There is a natural action of $R_{S,p}$ on the subspace $V_\pi^{K^S.I_n}$. The choice of an eigenspace is called a p-stabilization of π. Given an eigenspace, we write λ_π for the corresponding character

[2] It is easily checked that this algebra is independant of m.

of $R_{S,p}$. Of course, the choice of a p-stabilization is purely local at p: it depends only on the choice of an eigenvector for $\mathcal{U}_p$ in $\pi_p^{I_m}$.

For any $\tau = (c_d, ..., c_{b+1}; c_1, ..., c_b)$ as in §1.1, the associated normalized weight is $w_\tau := (c_1 - a, \dots, c_b - a, c_{b+1} + b, \dots c_d + b)$; this defines a dominant weight of the diagonal torus of $\mathrm{GL}_d(\mathbf{Q}_p)$ since $c_b - c_{b+1} \geq d$. For the purpose of p-adic variation we normalize the character λ_π, setting $\lambda_\pi^\dagger(f) = \lambda_\pi(f)$ for any $f \in \mathcal{C}_c^\infty(K_m^S \backslash G(\mathbf{A}_f^S)/K_m^S), \mathbf{Z})$ and

$$\lambda_\pi^\dagger(u_t) := \frac{\lambda_\pi(u_t)}{w_\tau(t)}$$

for any $u_t \in \mathcal{U}_p$. It can be checked (cf. [Hi04]) that this normalization preserves the p-integrality of the eigenvalues.

Given a p-stabilization of π, we let I_π be the distribution defined by

$$\mathcal{C}_c^\infty(G(\mathbf{A}_f^p), \mathbf{Z}) \otimes \mathcal{U}_p \ni f \otimes u_t \mapsto I_\pi(f) := tr(\pi(f))\lambda_\pi(u_t),$$

and we define the normalized distribution $I_\pi^\dagger$ by replacing λ_π with $\lambda_\pi^\dagger$. We call $I_\pi^\dagger$ a p-stabilized distribution associated to π. Note that $I_\pi^\dagger|_{\mathcal{R}_{S,p}} = \lambda_\pi^\dagger$.

Let $T_d \subset GL_d$ be the diagonal torus and $B_d \subset GL_d$ the Borel subgroup of upper-triangular matrices. Assume that $\pi_p = I(\chi) := Ind_{B_d(\mathbf{Q}_p)}^{\mathrm{GL}_d(\mathbf{Q}_p)} \chi$ with χ an unramified character of $T_d(\mathbf{Q}_p)$. Let $I = I_1$. The choice of a p-stabilization is given by the choice of an eigenvector for $\mathcal{U}_p$ in $I(\chi)^I$. For each element of the Weyl group $W(G,T)$, there exists such an eigenvector $v_{\chi,w} \in I(\chi)^I$ with the property that

$$u_t.v_{\chi,w} = \chi^w \rho(t) \cdot v_{\chi,w},$$

where $\rho = (\frac{d-1}{2}, \frac{d-3}{2}, \dots, \frac{1-d}{2})$ is half the sum of the positive roots. The choice of a p-stabilization is therefore equivalent to an ordering of the Langlands parameters of the spherical representation $I(\chi)$. If $(\alpha_1, \dots, \alpha_d)$ is the corresponding ordering, then we have

$$\lambda_\pi(u_t) = \prod_{i=1}^d \alpha_i^{t_i}.$$

In general, if π_p is spherical but associated to a non-unitary character χ, it may not equal the full induction of χ, in which case some orderings of the Langlands parameters do not have a corresponding p-stabilizations (see [SU02] for an example in the symplectic case).

A p-stabilization is said to be of finite slope if there is a d-tuple $s(\lambda_\pi^\dagger) = (s_1, \dots, s_n) \in \mathbf{Q}^n$ such that

$$v_p(\lambda_\pi^\dagger(u_t)) = -\sum_{k=1}^d t_k.s_k, \quad t = \mathrm{diag}(t_1, \dots, t_d).$$

Such a d-tuple is necessarily unique and called the slope of the p-stabilization. The integrality of the normalization implies that $s(\lambda_\pi^\dagger)$ belongs to the positive obtuse cone (in more automorphic terms this means that the Newton polygon lies above the Hodge polygon), and it can be easily checked that $s_1 + \cdots + s_d = 0$ (i.e., the Newton and Hodge polygon meet at their beginning and end) by considering the action of the center. If $s(\lambda_\pi^\dagger) = (0, \ldots, 0)$ then the p-stabilization is said to be ordinary. In general, the slope is said to be non-critical if $s_{i+1} - s_i < c_{i+1} - c_i + 2$ for all $i = 1, \ldots, d-1$. Otherwise, it is said to be critical. Note that the non-critical conditions define an alcove of the obtuse cone.

2.3. **Families.** We consider $\mathfrak{X}_{/\mathbf{Q}_p}$, the rigid analytic variety over $\mathbf{Q}_p$ such that

$$\mathfrak{X}(L) = Hom_{cont}(T(\mathbf{Z}_p), L^\times)$$

for any finite extension L of $\mathbf{Q}_p$. A point (or p-adic weight) $w \in \mathfrak{X}(L)$ is called arithmetic if the restriction of w to some open subgroup of $T(\mathbf{Z}_p)$ is algebraic and dominant. The corresponding algebraic character is then denoted $w^{alg} = (a_1, \ldots, a_d)$ and we write $\mathfrak{X}(\overline{\mathbf{Q}}_p)^{alg}$ for the subset of arithmetic weights. We sometimes write $\mathfrak{X}^d$ instead of $\mathfrak{X}$, d being the dimension of $\mathfrak{X}$, to emphasize the dimension of $\mathfrak{X}$.

For any rigid space $\mathfrak{U}$ we denote by $A(\mathfrak{U})$ the ring of analytic function on $\mathfrak{U}$. For our purposes a p-adic family of automorphic forms is a character:

$$\lambda : \mathcal{R}_{S,p} \to A(\mathfrak{U}),$$

where $\mathfrak{U}$ is an irreducible rigid space over $\mathfrak{X}$ of positive dimension with projection map denoted $\mathbf{w}$ and such that there is a Zariski dense set of points $\Sigma \subset \mathfrak{U}(\overline{\mathbf{Q}}_p)^{alg} := \{y \in \mathfrak{U}(\overline{\mathbf{Q}}_p) | \ \mathbf{w}(y) \in \mathfrak{X}^{alg}(\overline{\mathbf{Q}}_p)\}$ with the property that for any $y \in \Sigma$ the compositum λ_y of λ with the evalutation map[3] at y is a normalized p-stabilization $\lambda_\pi^\dagger$ of an automorphic representation π of normalized weight $\mathbf{w}(y)^{alg}$.

Definition 2.3.1 (strong form)**.** *A p-adic family of automorphic representations is a linear functional*

$$I : \mathcal{C}_c^\infty(G(\mathbf{A}_S), \mathbf{Z}_p) \otimes \mathcal{R}_{S,p} \to A(\mathfrak{U})$$

such that $\lambda_I := I|_{\mathcal{R}_{S,p}}$ is a p-adic family in the weak sense and such that for all $y \in \Sigma$ as above, the compositum I_y of I with the evalutation map at y is the p-stabilized distribution $I_\pi^\dagger$ of an automorphic representation π of normalized weight $\mathbf{w}(y)^{alg}$.

Note that these definitions, and hence all subsequent discussion, are relative to some fixed set S of primes containing the prime p.

A fundamental question in area is whether an individual p-stabilization $I_\pi^\dagger$ is a member of a p-adic family. A positive answer to this question has been given by Hida in the ordinary case. Using the techniques of [Hi04], it can be shown that any ordinary p-adic *almost cuspidal*[4] eigenform is the member of a (strong) p-adic family of almost cuspidal eigenforms of dimension d. We say that an holomorphic modular form for $G_{a,b}$ is almost cuspidal if its constant terms along the parabolic subgroup $P_{a-1,b-1}$ are cuspidal. Using the techniques of [SU06], this can be generalized to forms with slope s statisfying $s_b = s_{b+1}$ (i.e., the semi-ordinary case, which means that $I_\pi^\dagger(u_0)$ is a p-adic

[3]The map $A(\mathfrak{U}) \to \overline{\mathbf{Q}}_p$ given by $f \mapsto f(y)$.

[4]This terminology is non standard.

unit where u_0 is the operator corresponding to the trace of the relative Frobenius [5] on the Shimura variety associated to $G_{a,b}$ in characteristic p. Note also that the following theorem is a special case of the results of [U06].

Theorem 2.3.2. *If I_0 is a non-critical cuspidal finite slope distribution of regular arithmetic weight w_0, then there exists $\mathfrak{U} \xrightarrow{\mathbf{w}} \mathfrak{X}$ of dimension d, $y_0 \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$, and a p-adic $\mathfrak{U}$-family I such that*

$$I_{y_0} = I_0 + I_1 + \cdots + I_s$$

with $I_1, \dots, I_s$ irreducible character distributions of $C_c^\infty(\mathbf{A}_S) \otimes \mathcal{R}_{S,p}$ such that

$$I_i|_{\mathcal{R}_{S,p}} = I_0|_{\mathcal{R}_{S,p}} \qquad \forall i = 1, \dots, s.$$

We expect that a similar result must be true for general *overconvergent* modular forms. Using techniques from Kisin-Lai [KL], it is possible to construct such a deformation provided one only requires it to be of dimension one. We will use this technique for critical Eisenstein series.

3. Deformations of Eisenstein series

We keep to the notation of sections 1 and 2. Recall that we have groups $G = G_{a,b}$ and $H = G_{a',b'}$ and $L = G \times Res_{\mathcal{K}/\mathbf{Q}}\mathbf{G}_m$ a standard Levi subgroup of a parabolic P of H. In this sectiopn, we will consider specific p-adic families for the groups G and H. Keeping with our practice from section 1, we will add a superscript $'$ when the notion is relative to H. For instance, I'_m means an Iwahori subgroup of $H(\mathbf{Q}_p)$.

3.1. **Critical Eisenstein series.** We now fix a cuspidal tempered representation π of $G(\mathbf{A})$ and an idele class character χ of $\mathbf{A}_{\mathcal{K}}^\times$ as in §1.4. We will assume that

$$\chi' = |\cdot|_{\mathbf{A_Q}}^{2\kappa'} \eta_{\mathcal{K}}^d \tag{3.1.1}$$

and that the assumptions of Proposition 1.4.1 are satisfied along with

$$L(\pi, \chi^{-1}, \kappa' + 1/2) = 0. \tag{3.1.2}$$

To simplify matters, we will also assume that π and χ are unramified at primes above p. We let S be the set comprising the primes of ramification of π, χ, and G (and hence also of H) together with p. Let $m > 0$ be an integer. Then $\pi_p^{I_m} \neq 0$, and we choose $v_0 \in \pi_p^{I_m}$ a p-stabilization of π_p. We consider the section $\Phi_p^{crit} \in I(\rho_p)$ defined for all $h \in H(\mathbf{Q}_p)$ by

$$\Phi_p^{crit}(h) = \begin{cases} \chi(t)\pi_p(g).v_0 & \text{if } h = n.m(g,t)wk_0 \in P(\mathbf{Q}_p)wI'_m \\ 0 & \text{otherwise} \end{cases} \tag{3.1.3}$$

Here w is the Weyl element from (1.4.1).

[5]It corresponds to the d-tuplet $(\underbrace{1, \dots, 1}_{b}, \underbrace{p, \dots, p}_{a})$.

For $s \in \mathbf{C}$ let $I(\rho_v, s) = \{\varphi_s = \varphi\delta^{s+1/2}\ ;\ \varphi \in I(\rho_v)\}$. The following lemma follows from a direct computation.

Lemma 3.1.1. *For each $s \in \mathbf{C}$ the section $\Phi_{p,s}^{crit} = \Phi_p^{crit}\delta^{s+1/2} \in I(\rho_p, s)$ is an eigenvector for the action of $\mathcal{U}_p'$. Moreover, if $(\alpha_1, \dots, \alpha_d)$ is the ordering of Langlands parameters specifying the chosen p-stabilization v_0 of π_p then the ordering associated to Φ_{p,s_0}^{crit} is given by*

$$(\alpha_1, \dots, \alpha_b, \chi(\varpi)p^{\kappa'}, \chi(\varpi)p^{\kappa'+1}, \alpha_{b+1}, \dots, \alpha_d),$$

and if the slope of v_0 is $(s_1, \dots, s_d)$ then the slope of Φ_{p,s_0}^{crit} is

$$(s_1, \dots, s_b, 1, -1, s_{b+1}, \dots, s_d).$$

In particular, it is critical[6].

We consider the space V_0 generated by the Eisenstein series $E(\mathcal{F}_h; s_0, Z)$ associated to the sections $\Phi = \Phi_{\infty,i} \otimes \Phi_p^{crit} \otimes \Phi^{p,\infty}$ with $\Phi^{p,\infty} = \otimes_{v \neq p,\infty}\Phi_v$ with $\Phi_v = \Phi_v^{sph}$ if $v \notin S$. We let $V_1 \subset V_0$ be the subspace of Eisenstein series as above with the extra condition that $M(\Phi_v, s_0) = 0$ for all $v \in S\backslash\{p\}$ and let $E^{cr}(\pi, \chi) := V_0/V_1$. This last space, a quotient of a space of almost cuspidal holomorphic automorphic forms for H of weight $\xi = (c_d, \dots, c_{b+1}, \kappa - d/2 - 1; c_1, \dots, c_b, \kappa + d/2 + 1)$, is acted on by $\mathcal{R}_{S,p} \otimes C_c^\infty(H(\mathbf{A}_S), \mathbf{Z}_p)$ and decomposes as

$$E^{cr}(\pi, \chi) = \bigotimes_{v \in S\backslash\{p\}} \mathcal{L}(\pi_v, \chi_v, s_0)$$

with $\mathcal{L}(\pi_v, \chi_v, s_0)$ the Langlands quotient of $I(\rho_v, s_0)$. We denote by $I_{E^{cr}(\pi,\chi)}$ the corresponding distribution of $\mathcal{R}_{S,p} \otimes C_c^\infty(H(\mathbf{A}_S), \mathbf{Z}_p)$.

For any finite place v of $\mathcal{K}$ and any representation Π_v of $GL_n(\mathcal{K}_v)$, we denote by $\mathrm{r}ec(\Pi_v)$ the n-dimensional representation of the Weil-Deligne group associated to Π_v by the local Langlands correspondance as established by Harris-Taylor [HT01]. Then we have

$$\mathrm{rec}(BC(\mathcal{L}(\pi_v, \chi_v, s_0))) = \mathrm{rec}(BC(\pi)_v) \oplus \chi_v \circ \mathrm{A}rt_{\mathcal{K}_v}^{-1} \oplus \epsilon^{-1}\chi_v \circ \mathrm{A}rt_{\mathcal{K}_v}^{-1}$$

where $\mathrm{A}rt_{\mathcal{K}_v}$ stands for the Artin reciprocity map sending a uniformizer to a geometric Frobenius and where ϵ denotes the cyclotomic character.

3.2. **p-adic deformations.** Let $\mathfrak{X}^{d+2}/\mathbf{Q}_p$ be the weight space for H. For any $w_0 = (c_1, \dots, c_{d+2}) \in \mathfrak{X}^{d+2}$, we put

$$\mathfrak{X}_{w_0}^{d+2} = \{w = (e_1, \dots, e_{d+2}) \in \mathfrak{X}^{d+2} | e_i - e_{i+1} = c_i - c_{i+1} \forall i \neq b+1\}$$

This is clearly a two-dimensional closed subspace of $\mathfrak{X}^{d+2}$.

Theorem 3.2.1. *Let $w_0 = w_\xi = (c_1 - a - 1, \dots, c_b - a - 1, \kappa + (b-a)/2, \kappa + (b-a)/2, c_{b+1} + b + 1, \dots, c_d + b + 1)$. There exist an affinoid $\mathfrak{U}$ sitting over $\mathfrak{X}_{w_0}^{d+2}$, a point*

[6] In contrast, the semi-ordinary p-stabilization obtained by taking $\Phi_p^{ord} := M(\Phi_p^{crit,\vee}, -s_0)$, where $\Phi_p^{crit,\vee} \in I(\rho_p^\vee)$ is defined analogously to Φ_p^{crit}, has slope $(s_1, \dots, s_b, 0, 0, s_{b+1}, \dots, s_d)$.

$y_0 \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$ over w_0, and a two-dimensional family $I : C_c^\infty(H(\mathbf{A}_S), \mathbf{Z}_p) \otimes \mathcal{R}_{S,p} \to A(\mathfrak{U})$ such that

$$I_{y_0} = I_{E^{cr}(\pi,\chi)} + I_1 + \cdots + I_s$$

with the I_i's irreducible characters distributions satisfying

$$I_i|_{\mathcal{R}_{S,p}} = I_{E^{cr}(\pi,\chi)}|_{\mathcal{R}_{S,p}} \qquad \forall i = 1, \ldots, s.$$

If $\pi_\infty = \pi_\tau^H$ with τ regular, then this family extends to a $d+2$-dimensional family over $\mathfrak{X}^{d+2}$.

Proof. We give just an idea of how this theorem is proved. The details will appear elsewhere. The proof does not require one to start from an Eisenstein series. The techniques one uses to prove the first point of this theorem are similar to those used by Coleman, Kisin-Lai, and Kassaei. The deformations are constructed by studying the compact action of u_0 on the space of overconvergent modular forms for H obtained by multiplying one of the original critical Eisenstein series by powers of a characteristic zero lifting of powers of the Hasse invariant[7]. This requires that we first establish the rationality of (scalar multiples of) our critical Eisenstein series. The proof then employs the theory of the canonical subgroup as developed by various authors (Abbes-Mokrane, Kisin-Lai, Conrad). This provides a one-variable family. To obtain a two-variable, one twists the one-variable family by anticyclotomic characters of p-power conductor. To prove the second point, one shows that the constructed curve sits in the eigenvarieties associated to H in [U06]. The regularity condition on τ should not be necessary in this special case. In general, however, it might be necessary to make sure that the 'classical' systems of Hecke eigenvalues occuring in the one variable family contribute only to the middle cohomology of the Shimura variety for H.■

The next lemma helps describe the restrictions $I|_{C_c^\infty(H(\mathbf{Q}_v))}$, $v \in S\backslash\{p\}$. Its proof will appear elsewhere.

Lemma 3.2.2. *Let π_0 be a unitary irreducible representation of $G(\mathbf{Q}_v)$ and χ_0 a unitary character of $\mathcal{K}_v^\times$. Let $J : C_c^\infty(H(\mathbf{Q}_v)) \to A(\mathfrak{U})$ be an analytic $\mathfrak{U}$-family of local character distributions such that*

$$J_{x_0}(f) = tr(\mathcal{L}(\pi_0, \chi_0, s_0)(f)) + I_1(f) + \cdots + I_s(f)$$

where $I_1, \ldots, I_s$ are irreducible character distributions of $H(\mathbf{Q}_v)$. Assume J is generically irreducible. Then one of the two following cases holds:

- (i) *There exist an analytic $\mathfrak{U}$-family of representations π of $G(\mathbf{Q}_v)$ and an analytic $\mathfrak{U}$-family of characters χ of $\mathcal{K}_v$ such that $J_x(f) = tr(L(\pi_x, \chi_x, s_0)(f))$ for all $x \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$.*
- (ii) *The place v is split. There exist an analytic $\mathfrak{U}$-family of representations π of $GL_d(\mathbf{Q}_v)$ and two analytic $\mathfrak{U}$-families of characters μ and ν of $\mathbf{Q}_v$ with $\mu \neq \nu|\cdot|_v^{\pm 1}$*

[7]This is possible thanks to the theory of arithmetic toroidal compactification of the Shimura variety associated to H by K. Fujiwara [Fu]

such that $J_x^H(f) = tr((\mu_x \times \pi_x \times \nu_x)(f))$ *for a Zariski dense set of points* $x \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$ *where* $\mu_x \times \pi_x \times \nu_x$ *is the irreducible induction* $Ind_{\mathbf{G}_m \times GL_d \times \mathbf{G}_m}^{GL_{d+2}} \mu_x \otimes \pi_x \otimes \nu_x$.

4. Galois representations and applications to Selmer groups

4.1. **Galois representations for automorphic representations.** We begin with notation for the local theory.

Let w be a finite place of $\mathcal{K}$ and $G_{\mathcal{K}_w}$ the absolute Galois group of the completion of $\mathcal{K}$ at w. We denote by $Frob_w \in G_{\mathcal{K}_w}$ a geometric Frobenius element, $I_{\mathcal{K}_w} \subset G_{\mathcal{K}_w}$ the inertia subgroup, and $W_{\mathcal{K}_w} \subset G_{\mathcal{K}_w}$ the Weil subgroup.

Assume first that the residual characteristic of w is not p. To any finite-dimensional representation $R : G_{\mathcal{K}_w} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$, one associates a Weil-Deligne representation $WD(R) = (r, N)$ where $r : W_{\mathcal{K}_w} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is a representation and $N \in M_n(\overline{\mathbf{Q}}_p)$ is such that

$$R(Frob_w^m \sigma) = r(Frob_w^m \sigma) exp(t(\sigma)N)$$

where $t : I_{\mathcal{K}_w} \to \mathbf{Z}_p$ is defined by $\sigma(\sqrt[p^f]{\varpi_w}) = \zeta_{p^f}^{t(\sigma)}.\sqrt[p^f]{\varpi_w}$ for a fixed choice of a compatible system $\{\zeta_{p^f}\}$ of p-power roots of unity and a uniformizer ϖ_w of $\mathcal{K}_w$. It is well-known that (r, N) is uniquely defined up to isomorphism.

If the residual characteristic of w is equal to p, one generally uses Fontaine's rings to study the p-adic representations of $G_{\mathcal{K}_w}$. If V is such a representation, one defines $D_?(V) = (V \otimes_{\mathbf{Q}_p} B_?)^{G_{\mathcal{K}_w}}$ with $? = dR, cris$ or st, where B_{dR}, B_{cris} and B_{st} are the usual rings of p-adic periods introduced by Fontaine. We write $D_{dR}^i(V)$ for the i-th step of the Hodge filtration of $D_{dR}(V)$. We adopt the geometric conventions for the Frobenius and the Hodge-Tate weights (so the Hodge-Tate weights of V are the jumps of the Hodge filtration of $D_{dR}(V)$).

In both the local and global cases, we denote by ϵ_p the p-adic cyclotomic character and we write $V(n)$ for the n-th Tate twist of a Galois representation V.

Let now $\pi = \pi_f \otimes \pi_\infty$ be an automorphic representation of $G(\mathbf{A})$ such that $\pi_\infty = \pi_\tau^H$ for some $\tau = (c_d, ..., c_{b+1}; c_1, ..., c_b)$. Let $\boldsymbol{\kappa}_\tau = (\kappa_1, ..., \kappa_d)$ be the strictly increasing sequence of integers defined by

$$\kappa_{d-i+1} := c_i + d - i + \delta_i \qquad \forall i = 1, \dots, d,$$

where $\delta_i = -a$ if $i \leq b$ and $\delta_i = b$ if $i \geq b+1$. Let S_π be the set of finite places of $\mathcal{K}$ above primes of ramification of π. The following conjecture[8] is expected to result from the stabilization of the trace formula for unitary groups.

Conjecture 4.1.1. *There exists a finite extension* L *of* $\overline{\mathbf{Q}}_p$ *and a Galois representation*

$$R_p(\pi) : G_{\mathcal{K}} \longrightarrow GL_d(L)$$

[8] This conjecture is a theorem for unitary groups appearing in the works of Kottwitz, Clozel, Harris-Taylor, Yoshida-Taylor [HT01, TY06]

satisfying the following properties:

1. $R_p(\pi)^\vee(1-d) \cong R_p(\pi)^c$
2. $R_p(\pi)$ *is unramified outside* $S_\pi \cup \{\wp, \bar{\wp}\}$
3. *For each finite place* w *of* $\mathcal{K}$ *of residue characteristic prime to* p*, we have*
$$WD(R_p(\pi)|_{W_{\mathcal{K}_w}}) \cong \mathrm{rec}(BC(\pi)_w^\vee \otimes |det|^{\frac{1-d}{2}})$$
where rec *is the reciprocity map given by the Local Langlands correspondance of Harris-Taylor* [HT01] *(using our identification of* $\mathbf{C}_p$ *with* $\mathbf{C}$*).*
4. $R_p(\pi)|_{G_{\mathcal{K}_\wp}}$ *is Hodge-Tate with Hodge-Tate weight given by* $\boldsymbol{\kappa}_\tau$*.*
5. *If* π_p *is unramified, the eigenvalues of the Frobenius endomorphism of* $D_{crys}(R_p(\pi))$ *are given by the Langlands parameters of* π_p *(again using the identification of* $\mathbf{C}_p$ *with* $\mathbf{C}$*)*

Let χ_p be the Galois character of $G_\mathcal{K}$ associated to an idele class character χ as in §1.4 (i.e., such that $\chi_p(Frob_w) = \chi(\varpi_w)$ if χ is unramified at w). We see in particular that (3) implies that

$$L^{\{p\}}(R_p(\pi)\otimes\chi_p, s) = L^{\{p\}}(\pi^\vee, \chi, s + \frac{1-d}{2}) \tag{4.1.1}$$

where $L^{\{p\}}$ means we have omitted the Euler factor at p and the L-function for the Galois representation is defined using the geometric Frobenius elements. Moreover, if χ also satisfies (3.1.1) then (4.1.1) implies

$$L^{(p)}(R_p(\pi)\otimes\chi_p, s) = L^{\{p\}}(\pi, \chi^{-1}, s + 2\kappa' + \frac{1-d}{2}). \tag{4.1.2}$$

4.2. **Families of Galois representations.** Let $\mathfrak{U}$ be a smooth connected affinoid variety defined over a $p-$adic field, and let G_F be the absolute Galois group of a number or ℓ-adic field F (ℓ may be equal to p). We call a pseudo-representation $T: G_F \to A^0(\mathfrak{U})$ an analytic family of Galois representations over $\mathfrak{U}$. For any reduced affinoid subdomain $\mathfrak{Z} \subset \mathfrak{U}$, we denote by $R_\mathfrak{Z}^T$ the semi-simple Galois representation (defined up to isomorphism) over a finite extension of the fraction ring $F(\mathfrak{Z})$ of $\mathfrak{Z}$ whose trace is the pseudo-representation $G_F \to A^0(\mathfrak{U}) \to A^0(\mathfrak{Z})$. We say that T is n-dimensional if $R_\mathfrak{Z}^T$ is. If $\mathcal{L}$ is a G_F-stable lattice of $R_\mathfrak{U}^T$ and $y \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$, then we denote by $R_y^\mathcal{L}$ the representation on the specialization $\mathcal{L}\otimes_{A_0(\mathfrak{U})} A(\mathfrak{U})/I_y$, where I_y is the ideal of analytic function on $\mathfrak{U}$ vanishing at y.

Assume F is a p-adic field. Let T be a family of representations of G_F of dimension d over an affinoid $\mathfrak{U}$. We denote by $\kappa_1, \dots, \kappa_d \in A(\mathfrak{U})$ the Hodge-Tate-Sen weights of T. Let r be the dimension of the affinoid $\mathfrak{U}$. The family T is said to be of finite slope[9] if there exist

(i) $\varphi_1, \dots, \varphi_d \in A^0(\mathfrak{U})$,
(ii) $\Sigma \subset \mathfrak{U}(\overline{\mathbf{Q}}_p)$,

[9]This is "trianguline" in the terminology of Colmez.

(iii) A subset of $\{1, \dots, d\}$ of r positive integers $i_1 \leq \dots \leq i_r$,

such that

(a) For all $y \in \Sigma$, we have the inequalities $\kappa_1(y) \leq \dots \leq \kappa_d(y)$,
(b1) For all $i \notin \{i_1, \dots, i_r\}$, $y \mapsto \kappa_{i+1}(y) - \kappa_i(y)$ is constant on $\mathfrak{U}$,
(b2) For all positive real numbers C, the subset Σ_C of points $y \in \Sigma$ such that $\kappa_{i_j}(y) - \kappa_{i_{j+1}}(y) > C$ for all $i = 1, \dots, r$ is Zariski dense.
(c) For all $y \in \Sigma$, R_y^T is crystalline, and the eigenvalues of Frobenius on $D_{cris}(R_y^T)$ are given by $\varphi_1(y)p^{\kappa_1(y)}, \dots, \varphi_d(y)p^{\kappa_d(y)}$

Let $y_0 \in \mathfrak{U}(L)$ such that $(\kappa_1(y_0), \dots, \kappa_d(y_0))$ is an increasing sequence of integers. According to a terminology of B. Mazur [M00], we say that T is a finite slope deformation of R_{y_0} of refinement $(\varphi_1(y_0)p^{k_1(y_0)}, \dots, \varphi_d(y_0)p^{k_d(y_0)})$ and Hodge-Tate variation $(i_1, i_2 - i_1, \dots, d - i_r)$.

Of course, there is a close link between p-stabilization and refinement. More precisely, we have the following easy lemma.

Lemma 4.2.1. *Assume conjecture 4.1.1. Let π be a cuspidal representation which is tempered and unramified at p and of weight τ (i.e., $\pi_\infty = \pi_\tau^H$). Let $\mathfrak{U}$ be an affinoid sitting over $\mathfrak{X}$ and let I be a p-adic deformation of π with p-stabilization given by $(\alpha_1, \dots, \alpha_d)$. Then there exists a p-adic deformation T_I over $\mathfrak{U}$ of $R_p(\pi)$ such that the restriction of T_I to $G_{\mathcal{K}_\wp}$ is a finite slope deformation of $\rho_\pi|_{G_{\mathcal{K}_\wp}}$ of refinement $(\varphi_1 p^{\kappa_1}, \dots, \varphi_d p^{\kappa_d})$ with*

$$\varphi_i = \alpha_{d-i+1}^{-1} . p^{-\kappa_i + (d-1)/2} \qquad \forall i = 1, \dots, d, \tag{4.2.1}$$

where $\boldsymbol{\kappa}_\tau = (\kappa_1, ..., \kappa_d)$.

Proof. The existence of T_I follows from the theory of pseudo-representations. The asserted properties of the restriction of T_I to $G_{\mathcal{K}_\wp}$ follows from parts (4) and (5) of the conjecture 4.1.1. The details are left to the reader.∎

We recall the following useful result of Kisin.

Proposition 4.2.2. *Let $T : G_F \to A^0(\mathfrak{U})$ be a finite slope family as above. Let $\mathcal{L}$ be any G_F-stable free $A(\mathfrak{U})$-lattice of $R_{\mathfrak{U}}^T$. Let F_0 be the maximal unramified subfield of F. After shrinking $\mathfrak{U}$ around some fixed $y_0 \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$, the following holds:*

(i) Let $1 \leq i \leq i_1$ be an an integer. If $y \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$ is such that $\kappa_i(y) \in \mathbf{Z}$, then

$$rk_{L \otimes K_0} D_{cris}(R_y^{\mathcal{L}})^{\phi = \varphi_i(y)p^{\kappa_i(y)}} \geq 1,$$

where $L = A(\mathfrak{U})/I_y$. Furthermore, there exists an integer N independant of y such that

$$D_{cris}(R_y^{\mathcal{L}})^{\phi = \varphi_i(y)p^{\kappa_i(y)}} \hookrightarrow (R_y^{\mathcal{L}} \otimes B_{dR}/t^{\kappa_i(y)+N} B_{dR}^+)^{G_K}.$$

for all $y \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$.

(ii) Let $Q_y(X) := \prod_{i=1}^{i_1}(X - \varphi_i(y)p^{\kappa_i(y)})$. *For any* y *such that* $\kappa_1(y) \in \mathbf{Z}$,

$$rk_{L\otimes F_0} D_{cris}(R_y^{\mathcal{L}})^{Q_y(\phi)=0} \geq i_1,$$

where $L = A(\mathfrak{U})/I_y$. *Furthermore, there exists an integer* N *independant of* y *such that*

$$D_{cris}(R_y^{\mathcal{L}})^{Q_y(\phi)=0} \hookrightarrow (R_y^{\mathcal{L}} \otimes B_{dR}/t^{\kappa_i(y)+N}B_{dR}^{+})^{G_K}.$$

for all $y \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$.

Proof. The first part of the proposition, and hence the second part when $Q_y(X)$ has only simple roots, is a direct consequence of Corollary 5.3 of [Ki]. When $Q_y(X)$ has multiple roots a simple generalization of the argument of [Ki] does the job. We can also as suggested to us by M.Kisin apply (i) to the case $\mathfrak{V} := Sp(A(\mathfrak{U})[X]/(Q(X))$ at least when $Q(X)$ has only generic simple roots.∎

We deduce from this proposition a few interesting consequences that we will use to construct elements in Selmer groups.

Lemma 4.2.3. *Let* T *be a finite slope* $\mathfrak{U}$*-family of representations of* G_F *as in Proposition 4.2.2, and let* $y \in \mathfrak{U}(L)$ *be such that* $R_y^T := L(1)^f \oplus L^e \oplus V^{ss}$ *for* V *a de Rham representation of* G_F. *We assume that*

(i) 1 *is a root of* $Q_y(X)$ *of order* e.
(ii) $D_{cris}(V)^{\phi=1} = 0$.

Let $\mathcal{L}$ *be a free lattice such that we have an exact sequence*

$$0 \to V \to R_y^{\mathcal{L}} \to W \to 0$$

then

(a) *Any non trivial extension of* L *by* $L(1)$ *appearing as a subquotient of* W *is crystalline.*
(b) *If* E *is the inverse image of* W^{G_F} *by the projection map from* $R_y^{\mathcal{L}}$ *to* W, *then* E *is an extension of* W^{G_F} *by* V, *the class* $[E]$ *of which is contained in* $H^1_f(K, Hom(W^{G_F}, V))$.

Proof. Since $D_{cris}(W/W^{G_F})^{\phi=1}$ and $D_{cris}(E)^{\phi=1}$ have rank at most $e - \dim W^{G_F}$ and $\dim W^{G_F}$, respectively, by hypothesis (ii), and since the rank of $D_{cris}(R_y^{\mathcal{L}})^{\phi=1}$ is e by hypothesis (i) and Proposition 4.2.2, we deduce that the respective ranks of $D_{cris}(W/W^{G_K})^{\phi=1}$ and $D_{cris}(E)^{\phi=1}$ equal $e-\dim W^{G_K}$ and $\dim W^{G_K}$. From $D_{cris}(V)^{\phi=1} = 0$ and $D_{cris}(E)^{\phi=1}$ being of rank $\dim W^{G_K}$, we deduce the surjectivity in the following short exact sequence:

$$0 \to D_{cris}(V) \to D_{cris}(E) \to D_{cris}(W^{G_K}) \to 0$$

The exactness of this sequence means, by definition, that $[E] \in H^1_f(K, Hom(W^{G_K}, V))$ and (b) is proved. The proof of (a) follows similarly using $\mathrm{rk} D_{cris}(W/W^{G_K})^{\phi=1} = e - \dim W^{G_K}$. The details are left to the reader.∎

The following lemma will be used in the last section of this paper.

Lemma 4.2.4. *Let K be a p-adic field and let R_0 be a de Rham representation of G_K over a finite extension L of $\mathbf{Q}_p$. Let $\mathfrak{U}$ be an affinoid and $T : G_K \to A(\mathfrak{U})$ a finite slope deformation of the character representation $T_0 = 1 + tr(R_0)$ of refinement $\varphi_1, \ldots, \varphi_{n+1}$ and Hodge-Tate weight variation $(i_1, i_2 - i_1, \ldots, n+1-i_r)$ Let $\mathcal{L}$ be a free G_K-stable $A(\mathfrak{U})$-lattice. We assume the following hypotheses are satisfied.*

(i) *$\varphi_i \neq 1$ if $i \leq i_1$.*
(ii) *There exists $y \in \mathfrak{U}(L)$ such that $T_y = T_0$ and $k_i(y) > 0$ for $i > i_1$.*
(iii) *The representation $R_{\mathfrak{U}}^{\mathcal{L}}$ is an extension of the form:*

$$0 \to A(\mathfrak{U}) \to R_{\mathfrak{U}}^{\mathcal{L}} \to S_{\mathfrak{U}} \to 0$$

with trivial action of G_K on $A(\mathfrak{U})$.

Then the rank over $L \otimes K$ of $gr^0 D_{dR}(R_y^{\mathcal{L}}) = (R_y^{\mathcal{L}} \otimes \mathbf{C}_p)^{G_K}$ is one more than the rank of $gr^0 D_{dR}(R_0)$

Proof. We have to prove that the Sen operator determining the action of a finite index subgroup of G_K on $R_y^{\mathcal{L}} \otimes \mathbf{C}_p$ has the eigenvalue 0 with multiplicity $1 + h_0$ with $h_0 :=$ rk $gr^0 D_{dR}(R_0)$. Equivalently, we need to show that the order of vanishing at 0 of the minimal polynomial of the Sen operator of $R_y^{\mathcal{L}}$ is one. By hypothesis (ii), it is easy to see that it is therefore sufficient to show the same statement for $R_z^{\mathcal{L}}$ for any z sufficiently closed to y and such that

(a) $k_i(z) = k_i(y)$ if $i \leq i_1$
(b) $k_i(z) > C$ if $i > i_1$

where C is any arbitrary large constant (we know that we can approach y by such points by the axioms of a finite slope deformation). We now prove the result for z satisfying (a) and (b).

After (if necessary) replacing $\mathfrak{U}$ by a sufficiently small neighborhood of y, we know by Proposition 4.2.2 that

$$D_{cris}(R_z^{\mathcal{L}})^{Q_z(\phi)=0} \otimes K \hookrightarrow (R_z^{\mathcal{L}} \otimes B_{dR}/t^{k_{i_1}(z)+N} B_{dR}^+)^{G_K}$$

If $C > N + k_{i_1}(y)$, we therefore have that if z statisfies (a) and (b) then the image of $D_{cris}(R_z^{\mathcal{L}})^{Q_z(\phi)=0} \otimes K \cap D_{dR}^0(R_z^{\mathcal{L}})$ in $gr^0(D_{dR}(R_z^{\mathcal{L}}))$ is of rank h_0. On the other hand, by our hypothesis (iii), we have an exact sequence

$$0 \to L \to R_z^{\mathcal{L}} \to S_z \to 0$$

and therefore $gr^0(D_{dR}(R_z^{\mathcal{L}}))$ contains also the non trivial image of $D_{cris}(L)$ on which the action of ϕ is given by the eigenvalue 1. By hypothesis (i) we may assume that $Q_z(1) \neq 0$ for z sufficiently close to y and therefore the images of $D_{cris}(L)$ and $D_{cris}(R_z^{\mathcal{L}})^{Q_z(\phi)=0} \otimes K \cap D_{dR}^0(R_z^{\mathcal{L}})$ in $gr^0(D_{dR}(R_z^{\mathcal{L}}))$ are disjoint and hence $gr^0(D_{dR}(R_z^{\mathcal{L}}))$ has rank $1 + h_0$. ∎

4.3. **Deformations of some reducible Galois representations and Selmer groups.** Let χ and π be as in §§1.4 and 3.1. Let S be a finite set of places of $\mathcal{K}$ containing $\wp, \wp^c$ and the primes of ramification of $BC(\pi)$ and χ. Assuming $L(\pi, \chi^{-1}, \kappa' + 1/2) = 0$, we have constructed in §§1 and 3 an Eisenstein representation $E^{cr}(\pi, \chi)$ whose S-primitive L-function is given by

$$L^S(E^{cr}(\pi, \chi), s) = L^S(\pi, s)L^S(\chi, s - \kappa' - 1/2)L^S((\chi^c)^{-1}, s + \kappa' + 1/2)$$

Therefore the Galois representation associated to our Eisenstein representation is:

$$R_p(\pi)(-1) \oplus \chi_p^{-1}\epsilon^{1+\kappa'-d/2} \oplus \chi_p^c\epsilon^{\kappa'-d/2}$$

Assume now that χ satisfies (3.1.1). We consider the Galois representation

$$R := R_p(\pi) \otimes \chi_p\epsilon^{d/2-\kappa'}.$$

It satisfies

$$R^c \cong R^\vee(1) \tag{4.3.1}$$

and we therefore have the functional equation

$$L(R, -s) = \epsilon(R, s)L(R, s),$$

and $s = 0$ is the central value for $L(R, s)$. By (4.1.2), $L^S(R, 0) = L^S(\pi, \chi^{-1}, \kappa' + 1/2)$. Note that the conditions on the weights of χ and π at the beginning of the section 1.4 implies that R does not have the Hodge-Tate weights 0 and -1. It can be seen that any (automorphic) irreducible Galois representation of $G_\mathcal{K}$ with regular Hodge-Tate weights having no Hodge-Tate weights equal to 0 and -1 and satisfying the condition (4.3.1) should be obtained in this way. Although it is not necessary, we will assume that R is irreducible.

The following result is suggested by the Bloch-Kato conjectures.

Theorem 4.3.1. *Assume conjecture 4.1.1 for unitary groups in $d+2$ variables. Assume π is tempered and that π and χ are unramified at primes above p. Assume also that $R_p(\pi)$ is irreducible. Then, if $L(R, 0) = 0$, we have*

$$\mathrm{rk}\ H^1_f(\mathcal{K}, R^\vee(1)) \geq 1$$

Here $H^1_f(\mathcal{K}, R^\vee(1))$ is the Bloch-Kato Selmer group associated to the p-adic representation $R^\vee(1)$; for a definition see [BK] or [FP].

Proof. The proof of this theorem runs along the same lines as that of Theorem 4.1.4 in [SU06].

We first choose a non-critical p-stabilization $(\alpha_1, \dots, \alpha_d)$ of π and denote by $(\varphi_1, \dots, \varphi_d)$ the corresponding refinement of $R_p(\pi)$ (given by (4.2.1)). Recall that we write τ for the weight of π_∞ and ξ for the weight of the Eisenstein series. Since we assume the existence of Galois representations for cuspidal representations of the unitary group $G_{a+1,b+1}$, by Theorem 3.2.1 there exists a two-dimensional affinoid subdomain $\mathfrak{U}$ sitting over a closed subspace of $\mathfrak{X}^{d+2}_{w_\xi}$, a point $y_0 \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$ over $w_0 = w_\xi$, and a $\mathfrak{U}$-family T of

Galois representations such that the specialization of T at y_0 is the pseudo-representation associated to $R_p(\pi)(-1) \oplus \chi_p^{-1}\epsilon^{\kappa'-d/2} \oplus \chi_p^{-1}\epsilon^{\kappa'-d/2-1}$ and such that the restriction of T to $G_{\mathcal{K}_\wp}$ is of refinement

$$(p\varphi_1, \ldots, p\varphi_a, \chi(\varpi)^{-1}p^{d/2-\kappa'+1}, \chi(\varpi)^{-1}p^{d/2-\kappa'}, p\varphi_{a+1}, \ldots, p\varphi_d)$$

and Hodge-Tate variation $(a+1, b+1)$ (i.e $r = 1$ and $i_1 = a+1$). Furthermore, from Lemma 3.2.2 and property (3) of the Conjecture 4.1.1, for all finite places w of $\mathcal{K}$ prime to p,

$$R^T_{\mathfrak{U}}|_{G_{\mathcal{K}_w}} \cong \mu_1 \oplus R_w \oplus \mu_2 \tag{4.3.1}$$

where μ_1, μ_2 are two $A(\mathfrak{U})$-valued characters of $G_{\mathcal{K}_w}$ specializing to $\chi_p^{-1}\epsilon^{\kappa'-d/2}|_{G_{\mathcal{K}_w}}$ and $\chi_p^{-1}\epsilon^{\kappa'-d/2+1}|_{G_{\mathcal{K}_w}}$ at the point y and R_w is a d-dimensional representation specializing to $R_p(\pi)|_{G_{\mathcal{K}_w}}$ at y.

We consider the normalized deformation $\tilde{R}_{\mathfrak{U}} := R^T_{\mathfrak{U}} \otimes \chi_p \epsilon_p^{d/2-\kappa'+1}$. We have $\tilde{R}^\vee_{\mathfrak{U}}(1) = \tilde{R}^c_{\mathfrak{U}}$, and the semi-simplified specialization of $\tilde{R}_{\mathfrak{U}}$ at $y \in \mathfrak{U}(L)$ is given by $\tilde{R}_{\mathfrak{U},y} = L \oplus L(1) \oplus R$. The restriction of $\tilde{R}_{\mathfrak{U}}$ to $G_{\mathcal{K}_\wp}$ is a deformation of $\tilde{R}_{\mathfrak{U},y}|_{G_{\mathcal{K}_\wp}}$ of refinement $(\beta_1, \ldots, \beta_a, 1, p^{-1}, \beta_{a+1}, \ldots, \beta_d)$ with $\beta_i = \varphi_i\chi(\varpi)p^{\kappa'-d/2}$ and of Hodge variation $(a+1, b+1)$. We deduce from this that the restriction of $\tilde{R}_{\mathfrak{U}}$ to the decomposition subgroup $G_{\mathcal{K}_{\wp^c}}$ is a deformation of $\tilde{R}_{\mathfrak{U},y}|_{G_{\mathcal{K}_{\wp^c}}}$ of refinement $(p^{-1}\beta_d^{-1}, \ldots, p^{-1}\beta_{a+1}^{-1}, 1, p^{-1}, p^{-1}\beta_a^{-1}, \ldots, p^{-1}\beta_1^{-1})$ and of Hodge variation $(b+1, a+1)$.

We claim that $Tr(\tilde{R}^{ss}_{\mathfrak{U}})$ is not of the form $T' + T''$ where T' and T'' are two pseudo-representations. Were this the case, then they would have to satisfy $T'_y(g) = 1 + \epsilon_p(g)$ and $T''(g) = tr(R(g))$ for all $g \in G_{\mathcal{K}}$. Assume this is so, and let us show we get a contradiction. First we show that the restriction to $G_{\mathcal{K}_\wp}$ of the representation R' associated to T' would be irreducible. By Proposition 4.2.2 the specialization of R' at any arithmetic point y' such that $s = \kappa_{b+2}(y') - \kappa_{b+1}(y') > 1$ would be a crystalline representation of Hodge-Tate weights $(0, s)$ and slopes $(1, s-1)$ and is therefore irreducible. The same statement holds for the restriction to $G_{\mathcal{K}_{\wp^c}}$. Moreover, the restriction of R' to $G_{\mathcal{K}_w}$ for $w \nmid p$ is a split sum of two characters by (4.3.1). Then exactly as in [SU06, Thms 4.2.7 or 4.3.4] we would deduce that there is an non-trivial extension class in $H^1_f(\mathcal{K}, \overline{\mathbf{Q}}_p(1))$; but we know that this group is trivial since the rank of the units in $\mathcal{K}$ is 0.

From the above discussion we deduce that $\tilde{R}_{\mathfrak{U}}$ is irreducible. Let $g \in G_{\mathcal{K}}$ be such that one of the eigenvalues, say α_0, of $\tilde{R}(g)$ is distinct from 1 and $\epsilon_p(g)$ and choose α in some finite normal extension $A(\mathfrak{V})$ of $A(\mathfrak{U})$ such that $\alpha(z) = \alpha_0$ for some $z \in \mathfrak{V}(\overline{\mathbf{Q}}_p)$ above y. We take v in the representation space of $\tilde{R}'_{\mathfrak{V}} := \tilde{R}'_{\mathfrak{U}} \otimes_{A(\mathfrak{U})} A(\mathfrak{V})$ such that $g.v = \alpha.v$. We then consider the $A(\mathfrak{V})$-lattice $\mathcal{L}$ of $\tilde{R}'_{\mathfrak{V}}$ generated by $g.v$ over $A(\mathfrak{V})$ as g runs through $G_{\mathcal{K}}$. After possibly shrinking $\mathfrak{V}$ around z, we can assume $\mathcal{L}$ is free. By construction, $\mathcal{L}_z$ has a unique irreducible quotient, and this quotient is isomorphic to R. We therefore have an exact sequence of $G_{\mathcal{K}}$-representations

$$0 \to W \to R^{\mathcal{L}}_z \to R \to 0$$

with $W^{ss} \cong L \oplus L(1)$, L being the residue field of z. We first note that by (4.3.1) the restriction of W to $G_{\mathcal{K}_w}$, $w \nmid p$, is split. Moreover, we know by the application of Proposition 4.2.2 that $D_{cris}(R_z^{\mathcal{L}})^{\phi=1}$ is non zero. Since 1 is not a root of Frobenius for R since $L(BC(\pi)_w, \chi_w, 1/2)^{-1} \neq 0$ at $w|p$ by [JS, sect. 2.5], we deduce $D_{cris}(W|_{G_{\mathcal{K}_w}})^{\phi=1}$ has rank 1 for all $w|p$. This shows that W is not a non-trivial extension of L by $L(1)$ since this extension would belong to $H^1_f(\mathcal{K}, L(1)) = 0$ (same argument as in [SU06, 4.3.4]).

Therefore L_z contains the trivial representation L and we can take $E := \mathcal{L}_z/L$. This gives a non-trivial extension:

$$0 \to R^\vee(1) \to E^\vee(1) \to L \to 0.$$

It follows from lemma 4.2.3, that $Res_{\mathcal{K}_w}([E^\vee(1)]) \in H^1_f(\mathcal{K}_w, R^\vee(1))$ for $w|p$. Note that we again use the fact that 1 is not a root of the Frobenius for $R^\vee(1) \cong R^c$ as this is an hypothesis of the quoted lemma. If $w \nmid p$, $Res_{\mathcal{K}_w}([E^\vee(1)]) \in H^1_f(\mathcal{K}_w, R^\vee(1))$ follows from (4.3.1). This ends the proof of the Theorem.■

5. Higher order vanishing and higher rank Selmer groups

5.1. **Higher order of vanishing.** In this section, we assume, as in Theorem 4.3.1, that

$$L(\pi, \chi^{-1}, 1/2 + \kappa') = L(R, 0) = 0$$

. Since we are assuming (4.3.1), the primitive L-function of the Eisenstein representation $E^{cr}(\pi, \chi)$ twisted by χ^{-1} is

$$L(E^{cr}(\pi, \chi), \chi^{-1}, s) = L(\pi, \chi^{-1}, s)\zeta_{\mathcal{K}}(s - \kappa' - 1/2)\zeta_{\mathcal{K}}(s - \kappa' + 1/2).$$

Therefore the order of vanishing of $L^S(E^{cr}(\pi, \chi), \chi^{-1}, s)$ at $s = s_0 = \kappa' + 1/2$ is one less than the order of vanishing of $L(\pi, \chi^{-1}, s)$ at $s = s_0$, because $\zeta_{\mathcal{K}}(0) \neq 0$ and $\zeta_{\mathcal{F}}(s)$ has a simple pole at $s = 1$. This remark is the starting point of a method of constructing a higher rank subspace in the Selmer group $H^1_f(\mathcal{K}, R^\vee(1))$ when such a space is predicted by the Bloch-Kato conjecture (i.e., when $L(R, s)$ vanishes to higher orders at $s = 0$). In this final section of this paper, we deal with the case of even order vanishing. More precisely, we will sketch a proof of the following theorem.

Theorem 5.1.1. *Let π and χ as in Theorem 4.3.1. We assume Conjecture 4.1.1 for $G_{a+2,b+2}$. If $L(R, s)$ vanishes to even order at $s = 0$, then*

$$\mathrm{rk}\ H^1_f(\mathcal{K}, R^\vee(1)) \geq 2$$

5.2. **Sketch of proof.** Since $L(R, s)$ vanishes to even order at $s = 0$, $\epsilon(\pi, \chi^{-1}, 1/2+\kappa') = \epsilon(R, 0) = 1$. This implies, by the remark at the begining of this section, that

$$\epsilon(R \oplus \varepsilon_p \oplus 1) = -\epsilon(R, 0) = -1. \tag{5.2.1}$$

Let $\mathfrak{U}$ above $\mathfrak{X}^{d+2}$ and Σ be as in Theorem 3.2.1 and let $\tilde{R}_{\mathfrak{U}}$ be as in the proof of Theorem 4.3.1. By (5.2.1), for each $z \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$, $\epsilon(\tilde{R}_z, 0) = -1$. In particular, for each arithmetic point $z \in \Sigma^{reg}$, the subset of elements $z \in \Sigma$ with $\mathbf{w}(z) = (w_1, ..., w_{d+2})$ satisfying the regularity condition $w_{b+1} \geq \kappa + (b-a)/2 + 1$ and $w_{b+2} \leq \kappa + (b-a)/2 + 1$ (the analog of (1.4.1) with d replaced by $d+2$), we have $L(\pi(z), \chi^{-1}, 1/2 + \kappa') = 0$ where $\pi(z)$ is the (holomorphic) cuspidal automorphic representation of $G_{a+1,b+1}(\mathbf{A})$ associated to z. We can therefore apply Proposition 1.4.1 and Theorem 3.2.1 with $\pi(z)$ and $G_{a+1,b+1}$ in place of π and $G_{a,b}$ and then repeat the argument of Theorem 4.3.1. Let ξ_z the weight of the Eisenstein series representation $E^{cr}(\pi(z), \chi)$. For each $z \in \Sigma$, there exists $\mathfrak{U}_z$ above $\mathfrak{X}^{d+4}_{w_{\xi_z}}$, a point $y_z \in \mathfrak{U}_z(\overline{\mathbf{Q}}_p)$ over w_{ξ_z}, and a pseudo-representation $T_z : G_{\mathcal{K}} \to A(\mathfrak{U}_z)$ as in the proof of Theorem 4.3.1.

Let w_1 be the arithmetic weight of $\mathfrak{X}^{d+4}$ defined by $w_1 := (c_1 - a - 2, \ldots, c_b - a - 2, \kappa + \frac{b-a}{2} - 1, \kappa + \frac{b-a}{2}, \kappa + \frac{b-a}{2}, \kappa + \frac{b-a}{2} + 1, c_{b+1} + b + 2, \ldots, c_d + b + 2)$. Let $\mathfrak{Y} \subset \mathfrak{X}^{d+4}$ be the set of weight $w = (e_1, \ldots, e_{d+4})$ such that $e_i = e_{i+1}$ for $i \neq b+1, b+2, b+3, d+4$. This is a 4-dimensional subspace of $\mathfrak{X}^{d+4}$. One can show there exists a 4-dimensional affinoid $\mathfrak{V}$ sitting over $\mathfrak{Y}_{w_1} := w_1 + \mathfrak{Y}$, containing $\mathfrak{U}_z$ for each $z \in \Sigma^{reg}$, and admitting a $\mathfrak{V}$-family of automorphic representations interpolating the $\mathfrak{U}_z$-families. In other words the $\mathfrak{U}_z$-families fit together into a 4-dimensional family.

Let $S : G_{\mathcal{K}} \to A(\mathfrak{V})$ be the Galois deformation associated to the above $\mathfrak{V}$-family. It is a deformation of

$$S_{y_1} = tr(R_p(\pi)(-2)) + \chi_p^{-1}\epsilon_p^{\kappa'-d/2-1} + \chi_p^{-1}\epsilon_p^{\kappa'-d/2-1} + \chi_p^{-1}\epsilon_p^{\kappa'-d/2-2} + \chi_p^{-1}\epsilon_p^{\kappa'-d/2-2}.$$

for some point $y_1 \in \mathfrak{V}(L)$ sitting over w_1. We consider the normalization defined by $\tilde{\tilde{R}}_{\mathfrak{V}} := R^S_{\mathfrak{V}} \otimes \chi_p \epsilon_p^{d/2-\kappa'+2}$. Then, $\tilde{\tilde{R}}_{\mathfrak{V}}$ is a deformation of $tr(R) + 1 + 1 + \epsilon_p + \epsilon_p$. It is also a deformation of $tr(R_z) + 1 + \epsilon_p$ for all $z \in \Sigma^{reg}$ where we have written $R_z := R_p(\pi(z)) \otimes \chi_p \epsilon_p^{d/2-\kappa'+1}$. From the construction, it follows also that $\tilde{\tilde{R}}_{\mathfrak{V}}|_{G_{\mathcal{K}_\wp}}$ is a finite slope deformation of refinement $(\beta_1, \ldots, \beta_a, 1, 1, p^{-1}, p^{-1}, \beta_{a+1}, \ldots, \beta_d)$ and Hodge variation type $(a+1, 1, 1, b+1)$, and similarly for the restriction of $\tilde{\tilde{R}}_{\mathfrak{V}}$ to $G_{\mathcal{K}_{\wp^c}}$.

As in the Theorem 4.3.1, we consider a lattice $\mathcal{L} \subset \tilde{\tilde{R}}_{\mathfrak{V}}$ such that the specialization $R^{\mathcal{L}}_{y_1}$ has a unique quotient isomorphic to R. In particular, this implies that $R^{\mathcal{L}}_z$ has a unique quotient isomorphic to R_z. Moreover, from the proof of the Theorem 4.3.1 applied to $\pi(z)$, we see that $R^{\mathcal{L}}_z$ contains the trivial representation as a unique subrepresentation and has a quotient defining an extension E_z whose class belongs to $H^1_f(\mathcal{K}, R_z^\vee(1))$. In what follows, we will assume for simplicity that $\mathcal{L}$ is free although this might not be the case in general. However, it would not be difficult - although a bit cumbersome - to put ourselves in such a situation with a 'localization' argument similar to the one used in [SU06, §4.3.2].

Let $\mathfrak{U} := \mathfrak{V} \times_{\mathfrak{Y}_{w_1}} \mathfrak{X}^{d+4}_{w_1}$. This is the Zariski closure of Σ^{reg}. It follows from the discussion above that $R^{\mathcal{L}}_{\mathfrak{V}} \otimes A(\mathfrak{U})$ contains the trivial representation and that the quotient $\tilde{E}$ by the latter is an extension

$$0 \to A(\mathfrak{U}).\epsilon_p \to \tilde{E} \to \tilde{R} \to 0$$

where $\tilde{R}$ is the deformation of $tr(R) + 1 + \epsilon_p$ having a unique quotient isomorphic to R that appeared in the proof of Theorem 4.3.1. Then the restriction of $\tilde{E}$ to $G_{\mathcal{K}_\wp}$ is a finite slope deformation of refinement $(\beta_1, \dots, \beta_a, 1, p^{-1}, p^{-1}, \beta_{a+1}, \dots, \beta_d)$ and Hodge variation type $(a+1, b+2)$, and similarly for the restriction of $\tilde{E}$ to $G_{\mathcal{K}_{\wp^c}}$.

We now study the specialization $\tilde{E}_{y_1}$. It has a unique quotient isomorphic to R and has semi-simplification $L \oplus L(1) \oplus L(1) \oplus R$. We first remark that the trivial representation has to be a subrepresentation of $\tilde{E}_{y_1}$, for otherwise the latter would contain a non-trivial extension of L by $L(1)$. This extension would be unramified outside p and crystalline at p by another application of Proposition 4.2.2 and therefore would give a non-trivial element in $H^1_f(\mathcal{K}, L(1))$.

Quotienting $\tilde{E}_{y_1}$ by this trivial representation, we get an extension E_1. We will now prove that E_1 contains $L(1) \oplus L(1)$. Otherwise, it will contain a non-trivial extension of $L(1)$ by $L(1)$. It is easy to see that this extension would be unramified outside p. It would also be Hodge-Tate by the Lemma 4.2.4 applied to $E_1 \otimes L(-1)$ with $R_0 = R(-1) \oplus L$. Such a non-trivial extension does not exist.

We deduce that $E_1^\vee(1)$ is an extension of the form:

$$0 \to R^\vee(1) \to E_1^\vee(1) \xrightarrow{f} V \to 0$$

with V a L-vector space of dimension 2 with trivial action of Galois. We deduce that we have an exact sequence

$$0 \to H^0(\mathcal{K}, R^\vee(1)) \to H^0(\mathcal{K}, E_1^\vee(1)) \to V \xrightarrow{\delta} H^1(\mathcal{K}, R^\vee(1)).$$

We have $H^0(\mathcal{K}, E_1^\vee(1)) = 0$, for otherwise $E_1^\vee(1)$ contains the trivial representation, but $R^\vee(1)$ is the only subrepresentation to $E^\vee(1)$ since R is the only quotient of E_1 and $R^\vee(1) \cong R^c$ does not contain the trivial representation by hypothesis. Thus δ is injective. We can show that its image is contained in $H^1_f(\mathcal{K}, R^\vee(1))$ using the Lemma 4.2.3 just as we proved this for the class $[E^\vee(1)]$ in the proof of Theorem 4.3.1. Since V is dimension 2, this proves the theorem.■

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