DELIGNE-LUSZTIG REPRESENTATIONS

WENQI LI

1. Fixing the Frobenius Problem

Recall from last time that for G a connected linear reductive algebraic group over $k = \overline{k}$ of characteristic p, we let X be the set of all Borel subgroups, which is then identified with G/B for any fixed Borel subgroup B, as G acts transitively on X by conjugation and the stabilizer (i.e. normalizer) of B is B itself. This is a flag variety (so it's projective). Also recall that W is the Weyl group and F is the Frobenius on G.

Fix a F-stable Borel subgroup B and its maximal torus T. We defined $X(w)$ for $w \in W$ as the set of Borel subgroups B' such that B' and $F(B')$ are in relative position w. We then defined $Y = G/U$ for U the unipotent radical of B, and

$$Y(w) = \{gU \mid g^{-1}F(g) \in UwU\}$$

and saw that $Y(w) \to X(w)$ is a T^{F_w}-torsor where $F_w = \mathrm{ad}(w) \circ F$ is the Frobenius F twisted by w. Now we will give a construction that produces a T^F-torsor that is isomorphic to $Y(w) \to X(w)$.

What's going on here is that a Borel subgroup B' is in $X(w)$ if and only if $B' = gBg^{-1}$ and $F(B') = gwBw^{-1}g^{-1}$ for some $g \in G$. Using the bijection $G/B \cong X$ given by $g \mapsto gBg^{-1}$, the g that gives rise to B' satisfies

$$F(g)BF(g)^{-1} = gwBw^{-1}g^{-1}.$$

This says $w^{-1}g^{-1}F(g)$ normalizes B, which happens if and only if $g^{-1}F(g) \in wB$. Thus, we see that the subset

$$\{g \in G \mid g^{-1}F(g) \in wB\}$$

parametrizes $X(w)$. Two elements g_1, g_2 in this set represent the same Borel subgroup in $X(w)$ if and only if $g_1Bg_1^{-1} = g_2Bg_2^{-1}$ and $g_1wBw^{-1}g_1^{-1} = g_2wBw^{-1}g_2^{-1}$, which translates to $g_2^{-1}g_1 \in B \cap wBw^{-1}$. Thus, an alternative description of $X(w)$ is

$$X(w) = \{g \in G \mid g^{-1}F(g) \in wB\}/(B \cap wBw^{-1}).$$

Using $B = TU$ and that w normalizes T, we have $B \cap wBw^{-1} = T(U \cap wUw^{-1})$. For each g with $g^{-1}F(g) \in wB$, we want to replace g by gt for some $t \in T$ such that $g^{-1}F(g) \in wU$. For which t would this work? We have

$$(gt)^{-1}F(gt) = t^{-1}g^{-1}F(g)F(t) \in t^{-1}wUF(t) = t^{-1}(wUw^{-1})(wF(t)w^{-1})w$$

Since T commutes with U and $T = wTw^{-1}$, which see that the above becomes

$$(wUw^{-1})(t^{-1}(wF(t)w^{-1}))w,$$

and this is in wU if and only if $t^{-1}(wF(t)w^{-1}) = 1$, i.e. $t \in T^{F_w}$. Thus, we get that

$$X(w) = \{g \in G \mid g^{-1}F(g) \in wU\}/T^{F_w}(U \cap wUw^{-1}).$$

A similar computation shows that

$$Y(w) = \{g \in G \mid g^{-1}F(g) \in wU\}/(U \cap wUw^{-1}).$$

Date: October 29, 2024.

To get rid of the twist by w, we define

Definition 1.1. Let T' be a F-stable maximal torus. Let B' be a Borel subgroup (no longer F-stable) that contains T with unipotent radical U'. Define

$$\begin{aligned} X_{T'\subset B'} &= \{g \in G \mid g^{-1}F(g) \in F(B')\}/(B' \cap F(B')) \\ &= \{g \in G \mid g^{-1}F(g) \in F(U')\}/T'^F(U' \cap F(U')). \end{aligned}$$

and

$$Y_{T'\subset B'} = \{g \in G \mid g^{-1}F(g) \in F(U')\}/(U' \cap F(U')).$$

Proposition 1.2. *Let T', B' be as above, but now assume $B' \in X(w)$, i.e. B' and $F(B')$ are in relative position w. We can choose $h \in G$ such that $h(T,B)h^{-1} = (T', B')$, so that the map $g \mapsto gh^{-1}$ gives an isomorphism from the T^{F_w}-torsor $Y_w \to X_w$ to the T'^F-torsor $Y_{T'\subset B'} \to X_{T'\subset B'}$.*

Proof. Replacing h by ht we may assume $h^{-1}F(h) = w$. If g is such that $g^{-1}F(g) \in wU$, then

$$(gh^{-1})^{-1}F(gh^{-1}) = hg^{-1}F(g)F(h)^{-1} \in hwUF(h)^{-1} = hwUw^{-1}h^{-1}hwF(h)^{-1}.$$

The relative position w condition means

$$F(B') = hwBw^{-1}h^{-1}.$$

Since T is F-stable and normalized by w, this means $F(U') = hwUw^{-1}h^{-1}$. Thus gh^{-1} satisfies the condition of being in $X_{T'\subset B'}$.

■

2. Representations

Recall from last time we defined the virtual representation

$$R_w^\theta = \sum_i (-1)^i H_c^i(Y(w), \overline{\mathbf{Q}}_l)[\theta].$$

We will prove that this is independent of w, and its character is independent of l.

Proposition 2.1. *Let X be a quasi-projective separated scheme that is finite type over an algebraically closed field of characteristic p, and let $\sigma : X \to X$ be an automorphism of finite order. Then*

$$\operatorname{tr}(\sigma^*, H_c^\bullet(X, \mathbf{Q}_l))$$

is an integer independent of l.

Here σ^* is induced effect of σ on cohomology, and $H_c^\bullet(X, \mathbf{Q}_l)$ is the direct sum of all degrees. For a map of graded vector space $f : V^\bullet \to V^\bullet$, its trace is defined as

$$\operatorname{tr}(f, V^\bullet) = \sum_i (-1)^i \operatorname{tr}(f, V^i).$$

Proof. The scheme X lives over some finite field $\mathbf{F}_q$, so let $F : X \to X$ be the Frobenius. For $n \geq 1$, the composition $F^n \circ \sigma$ is some other Frobenius, namely the one if we consider X as a scheme over $\mathbf{F}_{q^n}$. So the Lefschetz fixed point formula for Frobenius says that

$$\operatorname{tr}((F^n \circ \sigma)^*, H_c^\bullet(X, \mathbf{Q}_l)) = |X^{F^n \circ \sigma}|.$$

which is also the number of the $\mathbf{F}_{q^n}$ points.

On the other hand, since F^* and σ^* commutes (simultaneously diagonizable), and traces are sums of eigenvalues, the value $\mathrm{tr}((F^n \circ \sigma)^*, H_c^\bullet(X, \mathbf{Q}_l))$ as a function of n is of the form $\sum_{\lambda \in \overline{\mathbf{Q}_l}^*} a_\lambda \lambda^n$, where all but finitely many $a_\lambda = 0$. So we have

$$\sum_{\lambda \in \overline{\mathbf{Q}_l}^*} a_\lambda \lambda^n = |X^{F^n \circ \sigma}|.$$

The right side is an integer, so it is fixed by any automorphism of $\overline{\mathbf{Q}_l}$. Thus if τ is an automorphism of $\overline{\mathbf{Q}_l}$, we have

$$\sum_{\lambda \in \overline{\mathbf{Q}_l}^*} \tau(a_\lambda)\tau(\lambda)^n = \sum_{\lambda \in \overline{\mathbf{Q}_l}^*} a_\lambda \lambda^n.$$

The left side is equal to $\sum_{\lambda \in \overline{\mathbf{Q}_l}^*} \tau(a_{\tau^{-1}(\lambda)})\lambda^n$ after re-indexing, and one can check by using a Vandermonde determinant that $n \mapsto \lambda^n$ are linearly independent. So we conclude that

$$\tau(a_{\tau^{-1}(\lambda)}) = a_\lambda, \text{ or equivalently } \tau(a_\lambda) = a_{\tau(\lambda)}.$$

for all $\lambda \in \overline{\mathbf{Q}_l}$. Therefore $\mathrm{tr}(\sigma^*, H_c^\bullet(X, \mathbf{Q}_l)) = \sum_{\lambda \in \overline{\mathbf{Q}_l}^*} a_\lambda$ is fixed by any automorphism of $\overline{\mathbf{Q}_l}$, so it is in $\mathbf{Q}$. Since σ has finite order, the traces are sums of roots of unity, and hence algebraic, so being in $\mathbf{Q}$ implies that it is an integer. Also, these coefficients a_λ are independent of l since they are determined by the number of the $\mathbf{F}_{q^n}$ points of X. ■

For independence of w, we will focus on the $\theta = 1$ case, which is where the hard work really is. When $\theta = 1$, we have

$$R_w^1 = \sum_i (-1)^i H_c^i(X(w), \overline{\mathbf{Q}_l}).$$

For $w \in W$, we say an element is an F-conjugate of w if it is of the form $w_1 w F(w_1)^{-1}$ for some $w_1 \in W$.

Theorem 2.2. *R_w^1 depends only on the F-conjugacy class of w.*

To prove this theorem we will need to use the structure of the Weyl group as a Coxeter group. This means W is generated by elements $s_1, \cdots, s_n$ satisfying $(s_i s_j)^{m_{ij}} = 1$ where $m_{ii} = 1$ and $m_{ij} \geq 2$ (could be ∞). These generators are called fundamental reflections. For each $w \in W$, its length is denoted by $l(w)$, and it is the length of the minimal expression $w = s_{i_1} \cdots s_{i_k}$. Here is a lemma we need to use:

Lemma 2.3. *Let s, t be two fundamental reflections in W. For $w \in W$, if $l(swt) = l(w)$, then either $swt = w$ or $l(sw) = l(w) - 1$, or $l(wt) = l(w) - 1$.*

Proof of Theorem 2.2. Since the fundamental reflections generated the group, it suffices to consider w and $w' = swF(s)$ for some fundamental reflection s. Exchanging the roles of w and w', we may assume $l(w') \geq l(w)$. If $l(w') = l(w)$, the previous lemma implies that either $w = w'$ or $l(sw) = l(w) - 1$.

So there are only two cases to consider. The first one is $w = w_1 w_2$, and $w' = w_2 F(w_1)$, and $l(w) = l(w_1) + l(w_2) = l(w_2) + l(F(w_1)) = l(w')$. In this case, for any $B \in X(w)$, we have $B = gB_0 g^{-1}$ and

$$F(B) = g w_1 w_2 B_0 w_2^{-1} w_1^{-1} g^{-1}$$

So if we let $\sigma B = g w_1 B_0 w_1^{-1} g^{-1}$, then $(B, \sigma B) \in O(w_1)$, and $(\sigma B, F(B)) \in O(w_2)$.

Now σB is in $X(w')$, because $F(B)$ and $F(\sigma B)$ are in relative position $F(w_1)$, so σB and $F(\sigma B)$ are in relative position $w_2 F(w_1)$. This gives a map $\sigma : X(w) \to X(w')$.

The same procedure can be used on $w_2 F(w_1)$ and $F(w_1)F(w_2) = F(w)$, so we get a map $\tau : X(w') \to X(F(w))$. We have a commutative diagram

$$\begin{array}{ccc} X(w) & \xrightarrow{\sigma} & X(w') \\ {\scriptstyle F}\downarrow & \overset{\tau}{\swarrow} & \downarrow{\scriptstyle F} \\ X(F(w)) & \xrightarrow[\sigma^q]{} & X(F(w')) \end{array} .$$

Now by étale cohomology magic, F induces an equivalence of étale sites, so σ and τ also induce equivalences of étale sites. Thus we obtain an isomorphisms $H_c^i(X(w)) \xrightarrow{\sim} H_c^i(X(w'))$.

The second case is that $l(w') > l(w)$, and since $w' = swF(s)$ this means $l(w') = l(w) + 2$. Let $B \in X(w') = X(swF(s))$. Using the same method as before, we find γB and δB such that $(B, \gamma B) \in O(s)$, $(\gamma B, \delta B) \in O(w)$, and $(\delta B, F(B))$ is in $O(F(s))$.

We define

$$X_1 = \{B \in X(w') \mid \delta B = F(\gamma B)\} \text{ and } X_2 = \{B \in X(w') \mid \delta B \neq F(\gamma B)\}.$$

Then X_1 is a closed subset of $X(w')$ and X_2 is its open complement. This decomposition gives a long exact sequence

$$\cdots \to H_c^{i-1}(X_1) \to H_c^i(X_2) \to H_c^i(X(w')) \to H_c^i(X_1) \to \cdots$$

So for any $g \in G$, we have

$$\operatorname{tr}(g^*, H_c^\bullet(X(w'))) = \operatorname{tr}(g^*, H_c^\bullet(X_1)) + \operatorname{tr}(g^*, H_c^\bullet(X_2))$$

We will show the first term is $\operatorname{tr}(g^*, H_c^\bullet(X(w)))$ and the second term is 0. Then the representations R_w^1 and $R_{w'}^1$ have the same character, so they are isomorphic.

After decomposing, we obtain the map $\gamma : X_1 \to X(w)$. For each $B' \in X(w)$, the fiber $\gamma^{-1}(B')$ consists of Borel subgroups B such that $(B, B') \in O(s)$. This means the fiber $\gamma^{-1}(B')$ is an affine line over k (?), so X_1 is a fiber bundle with fibers being affine lines. Thus

$$H_c^i(X_1) \cong H_c^{i-2}(X(w))(-1).$$

This shows $\operatorname{tr}(g^*, H_c^\bullet(X_1)) = \operatorname{tr}(g^*, H_c^\bullet(X(w)))$.

A similar but slightly more complicated analysis for X_2 produces long exact sequence

$$\cdots H_c^{i-1}(X(sw)) \to H_c^i(X_2) \to H_c^{i-2}(X(sw))(-1) \to H_c^i(X(sw)) \to \cdots$$

which then implies $\operatorname{tr}(g^*, H_c^i(X_2)) = 0$. (The original DL paper gives a reference to Grothendieck.)

■

In terms of our new description $X_{T' \subset B'}$ and $Y_{T' \subset B'}$, we can define

Definition 2.4. For a character $\theta : T^F \to \overline{\mathbf{Q}_l}$, we define

$$R_{T' \subset B'}^\theta = \sum_i (-1)^i H_c^i(Y_{T' \subset B'})[\theta]$$

which is a virtual representation of G^F.

We have $R_{T'\subset B'} = R_w^{\theta\circ\mathrm{ad}(w)}$.

Corollary 2.5. *The virtual representation $R^1_{T'\subset B'}$ depends only on the G^F-conjugacy class of the maximal tori T'.*

Proof. We have $R^1_{T'\subset B'} = R^1_w$, which depends only on the F-conjugacy class of w in W by the previous theorem. This translates to dependence on the G^F-conjugacy class of T', because F-conjugacy classes in W parametrize G^F-conjugacy classes of F-stable maximal tori. ■

3. Green Functions and the Character Formula

Definition 3.1. Let T be an F-stable maximal torus of G. The Green function $Q_{T,G}(u)$ is the restriction to the unipotent elements in G^F of the character of the virtual representation $R^1_{T\subset B}$, where B is any Borel subgroup containing T.

By what's in the previous section, $Q_{T,G}(u)$ doesn't depend on B, so there is no subscript B in the notation.

We will prove (probably next time)

Theorem 3.2. *Let $x = su$ be the Jordan decomposition of $x \in G^F$ (so s is semisimple and u is unipotent). Then*

$$\mathrm{tr}(x, R^\theta_{T\subset B}) = \sum \frac{1}{|Z^0(s)^F|} \sum_{\substack{g\in G^F \\ \mathrm{ad}\, g(T)\subset Z^0(s)}} Q_{\mathrm{ad}\, g(T), Z^0(s)}(u)\, \mathrm{ad}\, g(\theta)(s).$$

This formula expresses the character of $R^\theta_{T\subset B}$ in terms of only θ and the some Green function, so it shows that $R^\theta_{T\subset B}$ is independent of the choice of B.