Group Actions on the Drinfeld Curve

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1 The Drinfeld Curve

$G := \mathrm{SL}_2(\mathbb{F}_q)$ for q a power of p a prime; $\mathbb{F} = \bar{\mathbb{F}}_q$.

Definition 1.1. The *Drinfeld curve* is the curve in $\mathbf{A}^2(\mathbb{F})$ defined by

$$\mathbf{Y} = \{(x, y) \in \mathbf{A}^2(\mathbb{F}) \mid xy^q - yx^q = 1\}$$

- Affine, smooth, irreducible
- Smooth because differential is $\begin{pmatrix} y^q & -x^q \end{pmatrix}$ (since we're in characteristic 0) which is nowhere 0.
- Irreducible because $XY^Q - YX^Q - 1$ is irreducible; you do a change of variables $Z = X/Y, T = 1/Y$, write this as $T^{Q+1} - Z^Q - Z$, and use Eisenstein criterion (observe that (Z) is a maximal ideal in $\mathbb{F}[Z]$
- Equipped with actions from G, μ_{q+1}, and the Frobenius F(linear action by G, scalar multiplication by μ_{q+1}, and raising to the power of q)
- Remark: the actions by G and μ_{q+1} are free, but their product is not. Indeed, $(-I, -1)\cdot(x, y) = (-1(-x + 0), -1(0 - y)) = (x, y)$.
- Note that an action by G permits an action by U (the unipotent matrices)
- $g \circ \zeta = \zeta \circ g$, $F \circ g = g \circ \zeta$, $F \circ \zeta = \zeta^{-1} \circ F$

Proposition 1. *If V and W are two irreducible varieties, $\varphi : V \to W$ a morphism between them, and Γ a finite group acting on V such that*

1. *φ is surjective,*
2. *$\varphi(v) = \varphi(v') \iff v = \gamma v'$ for some $\gamma \in \Gamma$,*
3. *There exists a regular value v_0 of V,*

Then φ induces an isomorphism between V/Γ and W.

- $\gamma : \mathbf{Y} \to \mathbf{A}^1(\mathbb{F})$, $(x, y) \mapsto xy^{q^2} - x^{q^2}y$; $\mu_{q+1} \rtimes F$ equivariant (for the action $\zeta \cdot z \mapsto \zeta^2 z$).
- $\rho : \mathbf{Y} \to \mathbf{A}^1(\mathbb{F}) \setminus \{0\}$, $(x, y) \mapsto y$, $\mu_{q+1} \rtimes F$ equivariant (for regular μ_{q+1}) action

- $\pi : \mathbf{Y} \to \mathbf{P}^1(\mathbb{F}) \setminus \mathbf{P}^1(\mathbb{F}_q)$, $(x, y) \mapsto [x : y]$, $G \times F$ equivariant

Proposition 2. *$\hat{\gamma} : \boldsymbol{Y}/G \to \boldsymbol{A}^1(\mathbb{F})$, $\hat{\rho} : \boldsymbol{Y}/U \to \boldsymbol{A}^1(\mathbb{F})$ (where U is the subgroup of unipotent upper triangular matrices), and $\hat{\pi} : \boldsymbol{Y}/\mu_{q+1} \to \boldsymbol{P}^1(\mathbb{F}) \setminus \boldsymbol{P}^1(\mathbb{F}_q)$ are all isomorphisms of varieties.*

Proof. (for π.) First, observe that $\pi(g(x, y)) = \pi(ax + by, cx + dy) = [ax + by : cx + dy] = g[x : y] = g(\pi(x, y))$ and $\pi(F(x, y)) = \pi(x^q, y^q) = [x^q : y^q] = F(\pi(x, y))$. We check that conditions $1, 2$, and 3 are satisfied by π.

1. Say we have some $[1 : a] \in \mathbf{P}^1(\mathbb{F})$, and take any lift of this to $\mathbf{A}^2(\mathbb{F})$; say, $(1, a)$. Let $r = a^q - a$. Since $\mathbb{F}$ is algebraically closed, $\exists u \in \mathbb{F}$ such that $u^{q+1} = r$. So

   $$\frac{a^q}{\lambda^{q+1}} - \frac{a}{\lambda^{q+1}} = 1 \implies (1/\lambda, a/\lambda) \in \mathbf{Y},$$

   and clearly $[1/\lambda : a/\lambda] = [1 : a]$.

   Don't need to be concerned about $r = 0$ because that would mean

   $$a = a^q \implies a \in \mathbb{F}_q.$$

2. Clear that $[x : y] = [\zeta x : \zeta y]$.

   If $(x, y), (a, b) \in \mathbf{Y}$, $[x : y] = [a : b]$, a.k.a. $a = \lambda x$ and $b = \lambda y$, then

   $$1 = \lambda^{q+1} x y^q - \lambda^{q+1} y x^q = \lambda^{q+1}(x y^q - y x^q) = \lambda^{q+1}$$

   $\implies \lambda \in \mu_{q+1}$

3. The differential is

   $$\begin{pmatrix} \frac{\partial \pi}{\partial x} & \frac{\partial \pi}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{1}{y} & \frac{-x}{y^2} \end{pmatrix},$$

which is nonzero everywhere on $\mathbf{Y}$, thus always surjective. ☐

Some fixed points results under certain Frobeniuses (i.e. endomorphisms F' such that $F'^n = F^n$ for some n):

- $(\mathbf{Y})^{\zeta F} = \varnothing$, by the above (i.e. because $(\mathbf{Y}/\mu_{q+1})^F = \varnothing$; if $[a^q : b^q] = [a : b]$, then that would imply $a, b \in \mathbb{F}_q$

Theorem 3. *For $\zeta \in \mu_{q+1}$,*

$$\#(\mathbf{Y}^{\zeta F^2}) = \begin{cases} 0 & \zeta \neq -1 \\ q^3 - q & \zeta = -1 \end{cases}$$

Proof. If $(x, y) \in \mathbf{Y}^{\zeta F^2}$ then $x = \zeta x^{q^2}, y = \zeta y^{q^2}$, so

$$1 = (x y^q - y x^q)^q = x^q y^{q^2} - y^q x^{q^2} = \zeta^{-1}(x^q y - y^q x) = -\zeta^{-1}$$

so we must have $\zeta^{-1} = -1 \implies \zeta = -1$ to have fixed points.

In this case, we want solutions (x, y) to

$$x = -x^{q^2},$$

$$xy^q - yx^q = 1,$$

and

$$y = -y^{q^2}.$$

Solutions to the first two force something to be a solution to the last one; we have

$$y^{q^2} = (\frac{1+yx^q}{x})^q = \frac{1+y^q x^{q^2}}{x^q} = \frac{1-xy^q}{x^q} = -y.$$

$q^2 - 1$ nonzero solutions to the first equation, and given an x, q solutions to the second equation. □

Remark. $\mathbf{Y}^{-F^2}$ has a single G-orbit, because G acts freely on $\mathbf{Y}$.

What's to come:

Definition 1.2. For θ a character of μ_{q+1}, V_θ a $K\mu_{q+1}$ (K=l-adic field with μ_{q+1} r.o.u.) module admitting that character, we define

$$R'(\theta) = -\sum_{i\geq 0}[H_c^i(\mathbf{Y}) \otimes_{K\mu_{q+1}} V_\theta]_G$$

where $[]_G$ is the character associated to a irreducible G-representation