Review of etale cohomology

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Let's recall something more familiar: coherent sheaf cohomology.

Let X be a variety and consider the Zariski topology on X. Then, sheaves will take open subsets of X to $\mathcal{O}(U)$-modules (abelian groups). Coherent cohomology is defined by taking the derived functor of global sections.

In etale cohomology, we consider $Y \to X$ that are etale, which means $Y \to X$ is smooth and of relative dimension zero. Sheaves will send etale morphisms to abelian groups. The global sections functor sends F to $F(\mathrm{Id}_X)$, and then we take the derived functor to get etale cohomology. If we apply this to the structure sheaf, this just recovers Zariski cohomology. If we take the sheaf $\mathbb{Q}_\ell$, we take some limit of sheaves on $\mathbb{Z}/\ell^n$ and invert ℓ.

Let's start with the case of a finite ring A and let $\mathsf{Mod}(X, A)$ denote A-module sheaves on X. As in Rafah's talk, we can take its derived category to obtain $D(X, A)$.

There are a number of important functors on $D(X, A)$:

Example 1. The global sections functor can then be extended to the derived category:

$F \in \mathsf{Mod}(X, A)$, whose global sections, i.e. $F(\mathrm{Id}_X)$, gives a functor $\Gamma\colon \mathsf{Mod}(X, A) \to \mathsf{Mod}_A$. This is left-exact, and hence we have an induced derived functor $R\Gamma\colon D(X, A) \to D(A)$. In particular, $H^i(X, F) = H^i(R\Gamma(F))$

Example 2. Let $f\colon X \to Y$ be a morphism of schemes. Then, there is a functor $f_*\colon \mathsf{Mod}(X, A) \to \mathsf{Mod}(Y, A)$ given by $f_* F(Y') = F(Y' \times_Y X)$, noting that $Y' \times_Y X \to X$ is etale.

This is left-exact, and hence we obtain an induced derived functor $Rf_*\colon D(X, A) \to D(Y, A)$.

Any such f_* admits a left adjoint f^* that is exact, which induces a derived functor $f^*\colon D(Y, A) \to D(X, A)$.

Example 3. Let $j\colon U \subset X$ be an open immersion. Then j^* moreover admits a left adjoint $j_!$ called the extension by zero, which satisfies the fact $j^* j_! = \mathrm{Id}$.

Hence, we have a triple of adjunctions $(j_!, j^*, j_*)$.

Let $i\colon Z \hookrightarrow X$ be the complement of U. Then i_* moreover admits a right adjoint $i^!$. Then, we have the following "excision sequences" in the derived category:

$$i_* Ri^! F \to F \to Rj_* j^* F \to \bullet,$$

$$j_!j^*F \to F \to i_*i^*F \to \bullet,$$

Everything works if A is replaced with $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}_\ell}$.

For non-proper varieties, we often work with compactly supported cohomology instead:

If $j\colon X \subset \overline{X}$ is an open immersion with $\overline{X}$ proper, then we define $H^i_c(X, F) \coloneqq H^i(\overline{X}, F)$ (this is independent of the choice of compactification—proper base change!). You can also think of this as $H^i(R\Gamma(j_!F))$.

A few facts:

Theorem 4. *Let X be a smooth variety of dimension d over an algebraically closed field k.*

(i) $H^i_c(X, A)$ is a finite-type A-module.

(ii) $H^i_c(X, A) = 0$ for $i < 0$ and $i > 2d$. Moreover, if X is affine, then $H^i_c(X, A) = 0$ for $i < d$.

(iii) If $d = 1$, n is invertible in k, and X is smooth and projective of genus g, we have

$$H^i(X, \mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } i = 0, \\ (\mathbb{Z}/n\mathbb{Z})^{2g} & \text{if } i = 1, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 2, \\ 0 & \text{else.} \end{cases}$$

Let us use the excision sequence:

Example 5. Let $U = X - D$, where X is a smooth projective curve over an algebraically closed field k and D is a finite collection of closed points. Let $i\colon D \hookrightarrow X$ and $j\colon U \subset X$. Let ℓ be invertible in k. We have

$$j_!j^*\mathbb{Q}_\ell \to \mathbb{Q}_\ell \to i_*i^*\mathbb{Q}_\ell \to \bullet,$$

which induces

$$0 \to H^0_c(U, \mathbb{Q}_\ell) \to H^0(X, \mathbb{Q}_\ell) \to H^0(D, \mathbb{Q}_\ell) \to H^1_c(U, \mathbb{Q}_\ell) \to$$

$$\to H^1(X, \mathbb{Q}_\ell) \to H^1_c(D, \mathbb{Q}_\ell) \to H^2_c(U, \mathbb{Q}_\ell) \to H^2(X, \mathbb{Q}_\ell) \to H^2_c(D, \mathbb{Q}_\ell) \to 0$$

Then, we have

$$0 \to H^0_c(U, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell \to \mathbb{Q}_\ell^{|D|} \to H^1_c(U, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell^{2g} \to 0 \to H^2_c(U, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell \to 0,$$

so

$$H^i_c(U, \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell^{2g+|D|-1} & \text{if } i = 1, \\ \mathbb{Q}_\ell & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Example 6. In particular, $H_c^2(\mathbb{A}^1, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ and $H_c^{i\neq 2}(\mathbb{A}^1, \mathbb{Q}_\ell) = 0$.

Note: I'm being sloppy (maybe on purpose)! You have to add twists.

The Grothendieck-Lefschetz fixed point theorem says the following:

Theorem 7. *If $X/\mathbb{F}_q$ is a variety, then*

$$\#X(\mathbb{F}_q) = \sum_i (-1)^i \operatorname{Tr}\left(\operatorname{Fr}_q : H_c^i\left(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell\right)\right)$$

($x \in X\left(\overline{\mathbb{F}_q}\right)$, then $x \in X(\mathbb{F}_q)$ iff $\operatorname{Fr}_q(x) = x$). Here, $\operatorname{Fr}_q\colon X \to X$ is the absolute Frobenius, which induces an action on $X_{\overline{\mathbb{F}_q}} \to X_{\overline{\mathbb{F}_q}}$, and hence $H_c^i\left(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell\right)$.

Let me quickly just say how Fr_q acts on $H_c^i\left(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell\right)$:

Consider the composition

$$\begin{aligned} R\Gamma_c\left(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell\right) &\cong R\Gamma\left(\overline{X}_{\overline{\mathbb{F}_q}}, j_!\mathbb{Q}_\ell\right) \\ &\to R\Gamma\left(\overline{X}_{\overline{\mathbb{F}_q}}, \operatorname{Fr}_q^* j_!\mathbb{Q}_\ell\right) \\ &\cong R\Gamma\left(\overline{X}_{\overline{\mathbb{F}_q}}, j_! \operatorname{Fr}_q^* \mathbb{Q}_\ell\right) \\ &\cong R\Gamma\left(\overline{X}_{\overline{\mathbb{F}_q}}, j_!\mathbb{Q}_\ell\right) \\ &\cong R\Gamma_c\left(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell\right). \end{aligned}$$

Then, apply cohomology to both sides.

We have an "ally" theorem of Deligne:

Theorem 8. *If $X/\mathbb{F}_q$ is a variety, then the eigenvalues of Fr_q on $H_c^i\left(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell\right)$ are algebraic numbers of size at most $q^{i/2}$.*

Deligne's theorem implies that $\operatorname{Tr}\left(\operatorname{Fr}_q, H_c^i\left(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell\right)\right) \le q^{i/2} \dim H_c^i\left(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell\right)$. So this implies that to calculate $\#X(\mathbb{F}_q)$, we need to (approximately) calculate the cohomology $H_c^i\left(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell\right)$ for i large and bound $\dim H_c^i\left(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell\right)$ for i small.

This is a general phenomenom in number theory, where we are able to reduce many questions to the case of a point-counting problem over some curve$/\mathbb{F}_q$; in these cases, there isn't much hope of computing all the cohomology groups directly, so it is crucial to be able to bound a large number of them.

Theorem 9. *If $X/\mathbb{F}_q$ is a variety, F a constructible ℓ-adic sheaf on X. Then*

$$\sum_{x \in X(\mathbb{F}_q)} \operatorname{Tr}\left(\operatorname{Fr}_q, F_{\overline{x}}\right) = \sum_i (-1)^i \operatorname{Tr}\left(\operatorname{Fr}_q, H_c^i\left(X_{\overline{\mathbb{F}_q}}, F\right)\right).$$

Deligne's Weil II is as follows:

Theorem 10. *The eigenvalues of* Fr_q *on* $H_c^i\left(X_{\overline{\mathbb{F}_q}}, F\right)$ *are bounded by* $q^{\frac{i+w}{2}}$ *if* F *is mixed of weight* $\leq w$.