Finite Groups of Lie Type

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1 Basics of Linear Algebraic Groups

Let G be a closed subgroup of $\mathrm{GL}_n(k)$ considered as an algebraic group, with $k = \overline{k}$.

Example. GL_n, SL_n, SO_n, etc.

We are interested in representations of these groups, and indeed each of them come with a given representation (adjoint). In the nicest imaginable representation representations, every matrix is simultaneously diagonalizable. But since that's not always possible, let's look at elements that behave nicely.

Definition 1.1. Say $g \in G$ is semi-simple if g is conjugate to a diagonal matrix.

What about elements that don't behave nicely? The furthest a matrix can be from diagonalizable is nilpotent. Can think of nilpotent matrices as conjugate to upper triangular matrices with zeroes on the diagonal.

Proposition 1.2 (Jordan Decomposition)**.** *Let $x \in G$. There are unique $n, s \in G$ with $x = s + n$, s semisimple, n nilpotent, and $[n, s] = Id$.*

But since our groups are multiplicative, we'd prefer a mutiplicative decomposition. Thus,

Definition 1.3. Say $u \in G$ is *unipotent* if $u - \mathrm{Id}$ is nilpotent.

Proposition 1.4 (Jordan Decomposition)**.** *Let $x \in G$. There are unique $u, s \in G$ with $x = un$, s semisimple, u unipotent, and $[u, s] = Id$.*

Now that we can split our group elements into good and bad parts, we want to control good and bad parts of the group itself. First, topologically:

Definition 1.5. Let G^0 be the maximal connected closed subgroup of G.

Thus, G/G^0 is a finite group.

Definition 1.6. G is solvable if there exists $1 = H_0 \lhd, ..., \lhd H_r = G$ with H_{i+1}/H_i equal $\mathbf{G}_a$ or $\mathbf{G}_m$.

Example. Upper triangular matrices, by peeling off elements.

Theorem 1.7 (Lie-Kolchin theorem). *If G is connected and solvable, every irreducible representation has dimension* 1*.*

Corollary 1.8. *Every connected solvable closed subgroup is conjugate to a subgroup of upper triangular matrices.*

Definition 1.9. The *radical* of G is the largest closed normal connected solvable subgroup. The *unipotent radical* of G is the largest closed normal unipotent subgroup.

Example. The radical of GL_n is scalar matrices. The radical of SL_n is trivial.

The unipotent radical of GL_n and SL_n are trivial.

Definition 1.10. Say G is *semi-simple* if it has trivial radical and *reductive* if it has trivial unipotent radical.

Note that the radical and unipotent radical are closed and normal, so we can quotient by them to get semi-simple and reductive groups.

Now that we know how to get rid of the bad parts of G, let's study the good parts.

Definition 1.11. A maximal closed solvable subgroup is a *Borel* subgroup. A torus is a subgroup isomorphic to $\mathbf{G}_m^r$.

Example. For $G = \mathrm{GL}_n$, we often fix the Borel B as upper triangular matrices and the maximal torus T as diagonal matrices.

Fact.

- All Borel subgroups and all maximal tori are conjugate.
- $N_G(B) = B$ and $C_G(T) = T$.

Definition 1.12. Fixing a Borel B and maximal torus $T \subset B$, the Weyl group $W := N_G(T)/T$.

Example. For $G = \mathrm{GL}_n$ and T diagonal matrices, $N_G(T)$ consists of monomial matrices (one entry per row and column). Quotienting by T yields permutation matrices, so $W = S_n$.

2 Finite Fields

Let $\mathbf{F}_q$ be a finite field, $k = \overline{\mathbf{F}}_q$. Our goal is to understand linear algebraic groups over the former using our understanding of those over the latter.

Last time we saw schemes over $\mathbf{F}_q$ can be base changed to schemes over k.

Let $G \leq \mathrm{GL}_n(k)$ be a linear algebraic group. We have $G = G_0 \otimes_{\mathbf{F}_q} k$, but G_0 is not unique!

Definition 2.1. A *standard Frobenius map* is:

$$F : G \to G, \quad (a_{ij}) \mapsto (a_{ij}^q)$$

A *Frobenius map* is $F' : G \to G$ with $(F')^m = F^m$ for F standard and $m \in \mathbf{N}$.

Fact. A choice F of Frobenius determines G_0 by $(X(k))^F = X_0(\mathbf{F}_q)$.

Example. Let $G = \mathrm{GL}_n(k)$. Let $F : (g_{ij}) \mapsto (g_{ij}^q)$. Then $G^F = \mathrm{GL}_n(\mathbf{F}_q)$.

Now let $F' : g \mapsto (F(g)^T)^{-1}$. Note $(F')^2 = F^2$. Then $G^{F'} = U_n(\mathbf{F}_q)$ since $F(g)^T g = \mathrm{Id}$.

Definition 2.2. The *Lang map* is $\mathcal{L} : G \to G$, $g \mapsto g^{-1}F(g)$.

Note $\ker \mathcal{L} = G^F$.

Theorem 2.3. *If G is connected then $\mathcal{L}$ is surjective.*

Proof sketch. For $x \in G$, define

$$\mathcal{L}_x : G \to G, \quad g \mapsto g^{-1}xF(g)$$

It suffices to show $\mathcal{L}_x$ is dominant (since then $\mathrm{im}(\mathcal{L}_x) \cap \mathrm{im}(\mathcal{L}) \neq \emptyset$, so $g^{-1}F(g) = h^{-1}xF(h)$ and $x = \mathcal{L}(gh^{-1})$).

Consider differential maps $T_G \to T_G$. Have $d\mathcal{L}_x = -1$ since $dF = 0$. So $\mathcal{L}_x$ is an immersion $G \to G$ (same dimension). Since G is connected (and smooth), $\mathcal{L}_x$ is dominant. □

Next we'll see how the Lang map helps us interpolate between k and $\mathbf{F}_q$.

Lemma 2.4. *Let G be connected acting on $X = X_0 \otimes_{\mathbf{F}_q} k$. Let $\mathcal{O}$ be an orbit such that $F(\mathcal{O}) \subset \mathcal{O}$ (i.e., $\mathcal{O}$ is F-*stable*). Then*

1. $\mathcal{O}^F \neq \emptyset$
2. *For $x \in \mathcal{O}^F$, $g \in g(x)$*

$$g(x) \in \mathcal{O}^F \iff \mathcal{L}(g) \in \mathrm{Stab}_G(x)$$

Proof sketch.

1. Let $x \in \mathcal{O}$. Then $F(x) = g(x)$ by stability. Let $g^{-1} = \mathcal{L}(h) = h^{-1}F(h)$ by surjectivity. Then

$$x = h^{-1}F(h)F(x) \iff hx = F(h)F(x) \iff hx = F(hx)$$

using compatibility of F actions on G and X.

2. $$\text{RHS} \iff (g^{-1}F(g))(x) = x \iff (F(g))(F(x)) = gx \iff F(gx) = gx$$

□

But we also want to work with G^F as well as X^F. To that end:

Definition 2.5. Say g, g' are *F-conjugate* if $\exists h \in G$ with $g' = hgF(h)^{-1}$.

Lemma 2.6. *Let G act on X and $\mathcal{O}$ be an F-stable orbit. For $x \in \mathcal{O}^F$ there is a bijection*

$$\{G^F\text{-orbits on } \mathcal{O}^F\} \xrightarrow{\cong} \{F\text{-conjugacy classes of } \mathrm{Stab}_G(x)/\mathrm{Stab}_G(x)^0\}$$

$gx \mapsto \mathcal{L}(g)$

Proof sketch. The F-conjugacy classes of G and G/G^0 are in bijection. □

Proposition 2.7. *There is a bijection*

$$\{G^F\text{-conjugacy classes of } x\} \xrightarrow{\cong} \{F\text{-conjugacy classes of } C_G(x)/C_G(x)^0\}$$

Proof. Let G act on itself via $g \cdot h = ghg^{-1}$. Then G^F-orbits are G^F-conjugacy classes and $\mathrm{Stab}_G(x) = C_G(x)$. □

Corollary 2.8. *There is a bijection*

$$\{G^F\text{-conjugacy classes of } F\text{-stable maximal tori}\} \xrightarrow{\cong} \{F\text{-conjugacy classes of } W\}$$

3 Bruhat Decomposition

Recall that Borel subgroups are "good." So let's learn more about them. Let G/k be a reductive connected algebraic group.

Fact. Let $\mathcal{B}$ be the set of all Borel subgroups of G. Then

$$G/B \xrightarrow{\cong} \mathcal{B}, \quad gB \mapsto gBg^{-1}$$

(recalling all $B \in \mathcal{B}$ are conjugate and $N_G(B) = B$).

Remark. Since B is not normal, G/B is not a group! It is, however, a projective variety. This makes B a *(minimal) parabolic subgroup.*

Example. For $G = \mathrm{GL}_n$, have G/B the *flag variety* parameterizing

$$V_1 \subset ... \subset V_n$$

with $\dim V_i = i$.

You can think of the data of a Borel subgroup as a fixed flag.

Now let $T \subset B$ be a maximal torus. Recall the Weyl group $N_G(T) = W$. It will have the following structure.

Definition 3.1. A *Coxeter group* is $\langle r_1, ..., r_n : (r_i r_j)^{m_{ij}} = 1 \rangle$ with $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$.

The *length* of an element r is the minimal ℓ such that $r = r_1 ... r_\ell$.

Fact. Weyl groups have the structure of Coxeter groups.

Example. For $G = \mathrm{SL}_n$, $W = S_n$. The reflections are simple transpositions $(k\ k+1)$.

Theorem 3.2. *Fix a Borel subgroup B and a maximal torus $T \subset B$. The corresponding* Bruhat decomposition *of G is*

$$G = \bigsqcup_{w \in W} B\dot{w}B$$

where $\dot{w} \in N_G(T)$ is a lift of w.

Example. Let $G = \mathrm{GL}_n$, B be upper triangular matrices, T be diagonal matrices, and represent W by permutation matrices.

Decomposition says every matrix A has unique representation $A = U_1 P U_2$.

Intuition: U_1 is row operations, U_2 is column operations, all down-right. Permutation matrix is left.

Corollary 3.3.

$$G/B = \bigsqcup_{w \in W} B\dot{w}B/B$$

and $B\dot{w}B/B \cong \mathbf{A}^{\ell(w)}$.

Example. Let $G = \mathrm{GL}_3$, B upper triangular matrices, T diagonal matrices. For $w = \mathrm{id}$, the stratum $B\dot{w}B$ consists of matrices like

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

and under the action of B there is a unique element, and indeed $\ell(w) = 0$. For $w = (1\ 2)$, the stratum $B\dot{w}B$ consists of matrices like

$$\begin{bmatrix} 0 & * & * \\ \boldsymbol{*} & * & * \\ 0 & 0 & * \end{bmatrix}$$

and the action of B on the right can't eliminate the bold $*$, so there is an $\mathbf{A}^1$ worth of elements, and indeed $\ell(w) = 1$.

Fact. $B\backslash G/B \xrightarrow{\simeq} G\backslash(G/B \times G/B)$ via $\overline{\overline{g}} \mapsto \overline{(\overline{1}, \overline{g})}$.

Finally, we see how we can use this description to understand W intrinsically.

Corollary 3.4. $W \xrightarrow{\simeq} \{G\text{-orbits on } \mathcal{B} \times \mathcal{B}\}$, $w \to \mathcal{O}(w)$.

Fact. Fixing a Borel subgroup B,

$$\mathcal{O}(w) \xrightarrow{\simeq} \{(g_1B, g_2B) : g_1^{-1}g_2 \in BwB\}$$

There is a fibration $\mathcal{O}(w) \to \mathcal{B}$ and over fixed B the fiber is BwB/B, so $\dim \mathcal{O}(w) = \dim(\mathcal{B}) + \ell(w)$.

To multiply $\mathcal{O}(w)$ and $\mathcal{O}(w')$, ensure $\ell(w) + \ell(w') = \ell(ww')$, and then

$$\mathcal{O}(w) \times_{\mathcal{B}} \mathcal{O}(w') \xrightarrow{\simeq} \mathcal{O}(ww')$$