Deligne-Lusztig Theory Seminar - Derived Categories

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1 Abelian Categories

Definition. We say that a category $\mathcal{C}$ is *additive* if it satisfies the following:

- For each $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has an abelian group stucture, and composition is bilinear.
- $\mathcal{C}$ has a zero object.
- $\mathcal{C}$ admits finite products and finite coproducts. (One implies the other)

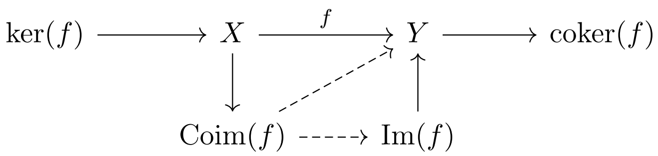
Definition. We say that an additive category $\mathcal{A}$ is *abelian* if it satisfies:

- $\mathcal{A}$ has all kernels and cokernels.
- For every morphism $f : X \to Y$ in $\mathcal{A}$, the map $\mathrm{Coim}(f) \xrightarrow{\sim} \mathrm{Im}(f)$ is an isomorphism.

Here the image and coimage of a map $f : X \to Y$ are defined as

$$\mathrm{Im}(f) = \ker(Y \to \mathrm{coker}(f)), \quad \mathrm{Coim}(f) = \mathrm{coker}(\ker(f) \to X),$$

and the map between them comes from the universal properties of kernel and cokernel applied to the diagram



Example. For the category $\mathbf{Mod}_R$ of modules over a ring R, this property becomes exactly the first isomorphism theorem, since for R-modules X, Y one has $\mathrm{Coim}(f) = X/\ker(f)$ and the corresponding morphism is $X/\ker(f) \xrightarrow{\sim} \mathrm{Im}(f)$. Therefore, $\mathbf{Mod}_R$ is an abelian category (all other properties are equally well known).

Remark. It is a well-known fact that all final limits can be recovered from the existence of an initial object, products and kernels (and analogously for finite colimits). Therefore, an abelian category admits all finite limits and colimits.

1.1 Complexes in abelian categories

Complexes can be defined in any additive category, but for the purposes of this talk we will assume we always work over an abelian category $\mathcal{A}$.

Definition. Let $\mathcal{A}$ be an abelian category. A (cochain) complex in $\mathcal{A}$ is a sequence

$$\cdots \to X^{n-1} \xrightarrow{\partial^{n-1}} X^n \xrightarrow{\partial^n} X^{n+1} \to \cdots$$

where $\delta^i \circ \delta^{i-1} = 0$ for all $i \in \mathbb{Z}$. The category of (cochain) complexes in $\mathcal{A}$ is denoted $C(\mathcal{A})$, with morphisms $f : X^\bullet \to Y^\bullet$ being commutative diagrams of morphisms $f : X^n \to Y^n$.

Definition. We define the *n-th cohomology* of a complex $X^\bullet$ to be the abelian group

$$H^n(X^\bullet) = \ker(\partial^n)/\mathrm{Im}(\partial^{n-1}).$$

This defines a functor $H^n : C(\mathcal{A}) \to \mathcal{A}$. We say that a morphism $f : X^\bullet \to Y^\bullet$ is a *quasi-isomorphism* if the induced maps in cohomology

$$H^n(f) : H^n(X^\bullet) \to H^n(Y^\bullet)$$

are isomorphisms.

Definition. We say that a morphism $f : X^\bullet \to Y^\bullet$ is *nullhomotopic* if there exist maps $h^n : X^n \to Y^{n-1}$ such that $f = h\delta_X + \delta_Y h$. We say that $X^\bullet \in C(\mathcal{A})$ is nullhomotopic if id_X is.

Definition. We define the homotopy category $K(\mathcal{A})$ to be the category whose objects are the same as $C(\mathcal{A})$, and the morphisms are morphisms of complexes modulo homotopy.

Warning. In general, the homotopy category $K(\mathcal{A})$ is **not** an abelian category. It has the structure of a triangulated category.

Remark. A nullhomotopic complex is quasi-isomorphic to zero, but the converse is false. In particular, taking cohomology induces functors $H^n : K(\mathcal{A}) \to \mathcal{A}$.

2 Derived Categories

Definition. The *derived category* $D(\mathcal{A})$ is a category equipped with a functor $Q : C(\mathcal{A}) \to D(\mathcal{A})$ such that Q sends quasi-isomorphisms to isomorphisms, and $(D(\mathcal{A}), Q)$ is initial for this property, meaning that for any category $\mathcal{B}$ and functor $Q' : C(\mathcal{A}) \to B$ that sends quasi-isomorphism to morphisms, Q' factors uniquely through Q.

The standard way to explicitly construct the derived category $D(\mathcal{A})$ is by localizing the homotopy category $K(\mathcal{A})$ at the quasi-isomorphisms. We will skip the details of how the localization works, but the idea is to define the full subcategory acyclic objects

$$N(\mathcal{A}) = \{X^\bullet \in K(\mathcal{A}) \mid H^k(X^\bullet) \simeq 0 \text{ for all } k\},$$

and formally invert the morphisms $X \to Y$ that embed into a distinguished triangle

$$X \to Y \to Z \to X[1]$$

with $Z \in N(A)$. We denote this localization by $K(\mathcal{A})/N(\mathcal{A})$, so this gives an explicit construction of $D(A)$.

Remark. Anything we have done for the category $C(\mathcal{A})$ can be analogously defined for the categories $C^+(\mathcal{A}), C^-(\mathcal{A})$ and $C^b(\mathcal{A})$ of bounded above, bounded below, and bounded (cochain) complexes. For $C^*(\mathcal{A})$ with $* = \varnothing, +, -, b$, we denote by $K^*(\mathcal{A})$, $N^*(\mathcal{A})$ and $D^*(\mathcal{A}) \simeq K^*(\mathcal{A})/N^*(\mathcal{A})$ the corresponding categories.

One interesting thing to note is that if $\mathcal{I}$ is a full additive subcategory of A that is cogenerating, meaning that for each $X \in \mathcal{A}$ there is $I \in \mathcal{I}$ and a monomorphism $0 \to X \to I$, then the functor $K^+(\mathcal{I}) \to D^+(\mathcal{A})$ is full and there is an equivalence $D^+(\mathcal{A}) \simeq K^+(I)/N^+(\mathcal{A})$. In particular, if $\mathcal{A}$ has enough injectives, the full subcategory $\mathcal{I}_A$ of injective objects in $\mathcal{A}$ and $N(\mathcal{I})$ has only the zero object, so we get an equivalence of categories

$$K^+(\mathcal{I}_A) \simeq D^+(\mathcal{A})$$

3 Derived Functors

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. Since nullhomotopy is a functorial property, F induces a functor $F : K(A) \to K(B)$ on homotopy categories. We want to find sufficient conditions for F to induce a functor on the derived categories.

Definition. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. We say that a class $\mathcal{I}$ of objects of $\mathcal{A}$ is F-injective if

- $\mathcal{I}$ is cogenerating.
- Given an exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$, if $X_1, X_2 \in \mathcal{I}$, then $X_3 \in \mathcal{I}$, and the sequence
$$0 \to F(X_1) \to F(X_2) \to F(X_2) \to 0$$
is exact.

Example. If $\mathcal{A}$ has enough injective objects (i.e. the class of injective objects is cogenerating), then the class of injective objects of $\mathcal{A}$ is F-injective for any left-exact functor $F : \mathcal{A} \to \mathcal{B}$.

Theorem 1. If $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor and there is an F-injective class $\mathcal{I}$ of objects of A, then F induces a functor
$$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B}),$$
called the *right derived functor* of F. Assume $\mathcal{A}$ admits enough injectives; then the cohomology of the complex $RF(X)$ coincides with the classical i-th derived functors:
$$H^i(RF(X)) \simeq R^i F(X) := H^i(F(I^\bullet)),$$
where $I^\bullet$ is an injective resolution of X.

Theorem 2. Let $F_1 : \mathcal{A}_1 \to \mathcal{A}_2$ and $F_2 : \mathcal{A}_2 \to \mathcal{A}_3$ be two functors such that there is an F_1-injective class $\mathcal{I}_1$ of objects in $\mathcal{A}_1$ and an F_2-injective class $\mathcal{I}_2$ of objects in $\mathcal{A}_2$ such that $F_1(\mathcal{I}_1) \subset \mathcal{I}_2$. Then $\mathcal{I}_1$ is $(F_2 \circ F_1)$-injective and
$$R(F_2 \circ F_1) \simeq RF_2 \circ RF_1$$

Remark. Everything in this section can be dualized so that if $G : \mathcal{A} \to \mathcal{B}$ is a right exact functor and the category has enough projectives, G defines a left derived functor
$$LG : D^-(\mathcal{A}) \to D^-(B)$$

3.1 Application to sheaves

The category $Sh_A(X)$ of sheaves of A-modules on a topological space X is an abelian category. A continuous map $f : X \to Y$ induces *pushforward* and *pullback* functors
$$f_* : Sh_A(X) \to Sh_A(Y), \quad f^* : Sh_A(Y) \to Sh_A(X),$$

defined as $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$, and f^* is given by the sheafification of the presheaf $U \mapsto \varinjlim_{f(U)\subseteq U'} \mathcal{F}(U')$. It is well known that the functor f_* is left exact, while the functor f^* is exact, and they are adjoint to each other. Since f_* is left exact, we can define the derived functor

$$Rf_* : D^+(Sh_A(X)) \to D^+(Sh_A(Y))$$

The pushforward commutes with compositions, i.e. $(f \circ g)_* = f_* \circ g_*$, and being the right adjoint of an exact functor, it preserves injectives. Therefore one gets

$$R(f \circ g)_* \simeq R(f_* \circ g_*) \simeq Rf_* \circ Rg_*$$

As a standard application, consider the final map $\sigma_X : X \to *$. The category of sheaves on $*$ is equivalent to the category of A-modules, and via this identification, the pushforward map $\sigma_{X,*} : Sh_A(X) \to \mathbf{Mod}_A$ can be identified with the global sections map $\Gamma(X, -)$. Note that for any map $f : X \to Y$ we have $\sigma_Y \circ f = \sigma_X$, so the previous result gives us the following isomorphisms in the derived categories (natural in $\mathcal{F}$)

$$R\Gamma(Y, Rf_*\mathcal{F}) \simeq R\Gamma(X, \mathcal{F})$$

Recall that the right derived functors of the global sections are the sheaf cohomology functors, this is $H^i(X, \mathcal{F}) := H^i R\Gamma(X, \mathcal{F})$. Therefore, taking cohomology, one recovers the Leray spectral sequence

$$H^p(Y, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

As a particular case, when f is a closed immersion, or when f is affine and $\mathcal{F}$ and we restrict to quasi-coherent sheaves, the pushforward f_* is exact, so $R^j f_* \simeq 0$ for $j > 0$ and one has

$$H^i(Y, f_*\mathcal{F}) \simeq H^i(X, \mathcal{F}).$$