Notes on Fourier Series and the Fourier Transform in $d > 1$

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1 Introduction

So far in this course we have been discussing Fourier analysis for functions of a single variable: functions on $\mathbf{R}$ in the Fourier transform case, periodic with period 2π in the Fourer series case. In this part of the course we'll first generalize to higher dimensions, then apply Fourier analysis techniques to study partial differential equations in higher dimensions. Unlike the last part of the course (distributions), the material we're covering here is generally well-described in the course textbook [1], with the notes here covering much the same material with less detail although a slightly different point of view (unlike [1], we'll sometimes work with distributions).

2 Fourier series and the Fourier transform for $d > 1$

2.1 Fourier series for $d > 1$

Consider a function $f(\theta_1, \theta_2, \cdots, \theta_d)$ of d variables, equivalently periodic on $\mathbf{R}^d$ with period 2π in each variable, or defined only for $-\pi < \theta_j \leq \pi$. We can think of such a function as defined on the circle S^1 in the case $d = 1$, in general on a product $S^1 \times \cdots \times S^1$ of d circles. We'll define Fourier series as the obvious generalization of the the $d = 1$ case. The Fourier coefficients of such a function will depend on integers $n_1, n_2, \cdots, n_d$ and be given by

$$\widehat{f}(n_1, n_2, \cdots, n_d) = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{-i(n_1\theta_1 + n_2\theta_2 + \cdots n_d\theta_d)} f(\theta_1, \theta_2, \cdots, \theta_d) d\theta_1 d\theta_2 \cdots \theta_d$$

Just like in the $d = 1$ case, some condition on the function f is needed that makes these integrals well defined.

One can ask whether Fourier inversion is true, in the sense that

$$f(\theta_1,\theta_2,\cdots,\theta_d)=\sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_d=-\infty}^{\infty}\widehat{f}(n_1,n_2,\cdots,n_d)e^{i(n_1\theta_1+n_2\theta_2+\cdots+n_d\theta_d)}$$

and there are similar theorems as in the d=1 case. If one considers just mean-square convergence, one finds that square-integrable functions satisfy Fourier inversion, and as Hilbert spaces

$$L^2(S^1\times\cdots\times S^1)=\ell^2(\mathbf{Z}^d)$$

(the Fourier transform preserves the inner product and the Parseval formula holds).

The situation with point-wise convergence is much worse in higher dimensions, since one has a d-fold infinite sum, with the sum in principle depending on the order of summation. However, the examples we saw of resummation and definitions of the sum that are given by limits of convolution with a "good" kernel still work fine. For instance, the obvious generalization of the heat kernel on the circle has the properties needed to ensure that

$$\lim_{t\to 0^+}(f*H_{t,S^1\times\cdots\times S^1})(\theta_1,\theta_2,\cdots,\theta_d)=f(\theta_1,\theta_2,\cdots,\theta_d)$$

2.2 The Fourier transform for $d>1$

The Fourier transform also generalizes in a straightforward way to $d>1$ dimensions. Using the vector notation

$$\mathbf{x}=(x_1,x_2,\cdots,x_d),\quad \mathbf{p}=(p_1,p_2,\cdots,p_d),\quad \mathbf{x}\cdot\mathbf{p}=x_1p_1+x_2p_2+\cdots x_dp_d$$

one defines

$$\widehat{f}(\mathbf{p})=\mathcal{F}f(\mathbf{p})=\int_{\mathbf{R}^d}f(\mathbf{x})e^{-2\pi i\mathbf{x}\cdot\mathbf{p}}dx_1\cdots dx_d$$

and would like to prove an inversion formula

$$f(\mathbf{x})=(\mathcal{F}^{-1}\widehat{f})(\mathbf{x})=\int_{\mathbf{R}^d}f(\mathbf{p})e^{2\pi i\mathbf{x}\cdot\mathbf{p}}dp_1\cdots dp_d$$

Just as in the Fourier series case, it turns out that if one just considers mean-squared convergence, using Lebesgue integration and appropriately defining the Fourier transform so as to give

$$\mathcal{F}:L^2(\mathbf{R}^d)\to L^2(\mathbf{R}^d)$$

then Fourier inversion holds ($\mathcal{F}^{-1}\mathcal{F}=1$) and $\mathcal{F}$ is an isomorphism of Hilbert spaces, so preserves inner products (Plancherel theorem), in particular

$$\int_{\mathbf{R}^d}|f(\mathbf{x})|^2dx_1dx_2\cdots dx_d=\int_{\mathbf{R}^d}|\widehat{f}(\mathbf{p})|^2dp_1dp_2\cdots dp_d$$

As in the $d = 1$ case, we'll proceed by first working with a special class of well-behaved functions, the Schwartz functions. In words the definition in higher dimensions is the same as in $d = 1$: a function f is in the Schwartz space $\mathcal{S}(\mathbf{R}^d)$ if f is smooth (C^∞) and f and all its derivatives fall off at $\pm\infty$ faster than any power. See the textbook [1], page 180 for a slightly more precise version of $\mathcal{S}(\mathbf{R}^d)$. We'lll treat less well-behaved functions as distributions, in a space $\mathcal{S}'(\mathbf{R}^d)$ of linear functionals on $\mathcal{S}(\mathbf{R}^d)$.

Many of the properties of the Fourier transform in $d > 1$ are much the same as in $d = 1$ and proven by essentially the same arguments. In particular

- The Fourier transform of a function in $\mathcal{S}(\mathbf{R}^d)$ is in $\mathcal{S}(\mathbf{R}^d)$.
- The Fourier transform on distributions in $\mathcal{S}'(\mathbf{R}^d)$ is defined as the transpose of the Fourier transform on functions in $\mathcal{S}(\mathbf{R}^d)$ and takes distributions to distributions.
- Fourier transformation takes translation by a vector $\mathbf{a}$ to multiplication by the function $e^{2\pi i\mathbf{p}\cdot\mathbf{a}}$.
- Fourier transformation takes the partial derivatives $\boldsymbol{\nabla} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_d})$ to multiplication by $2\pi i\mathbf{p}$.

3 Rotations and the Fourier transform

In any dimension, one can define rotations as those linear transformations that preserve the inner product:

Definition. *A rotation of* $\mathbf{R}^d$ *is a linear map*

$$R : \mathbf{x} \in \mathbf{R}^d \to R\mathbf{x} \in \mathbf{R}^d$$

such that

$$R\mathbf{x} \cdot R\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$$

These are the linear transformations that preserve lengths ($|\mathbf{x}|^2 = |R\mathbf{x}|^2$) and angles. Such linear transformations form a group, meaning

- The composition of two rotations is a rotation.
- There is a unit, the identity transformation I.
- Any rotation R has an inverse R^{-1} such that R composed with R^{-1} is the identity I.

The group of rotations in d dimensions is called $O(d)$, with the O for "orthogonal". A rotation R gives a linear transformation $\mathcal{R}$ on functions on $\mathbf{R}^d$

$$\mathcal{R} : f(\mathbf{x}) \to \mathcal{R}f = f(R\mathbf{x})$$

We define the action of rotations on distributions using the transpose, with an inverse so that the definition agrees with the definition on functions. If f is a distribution, then $\mathcal{R}f$ is the distribution given by

$$\langle \mathcal{R}f, \varphi \rangle = \langle f, \mathcal{R}^{-1}\varphi \rangle$$

For each rotation R we now have two linear transformations on $\mathcal{S}(\mathbf{R}^d)$ (and on $\mathcal{S}'(\mathbf{R}^d)$): the Fourier transform $\mathcal{F}$ and the rotation action $\mathcal{R}$. These commute

Claim. *For* $f \in \mathcal{S}(\mathbf{R}^d)$

$$\mathcal{R}\mathcal{F} = \mathcal{F}\mathcal{R}$$

Proof.

$$\begin{aligned}
\mathcal{R}\mathcal{F}f &= \widehat{f}(R\mathbf{p}) \\
&= \int_{\mathbf{R}^d} e^{2\pi i \mathbf{x}\cdot R\mathbf{p}} f(\mathbf{x}) dx_1 dx_2 \cdots dx_d \\
&= \int_{\mathbf{R}^d} e^{2\pi i (R\mathbf{x}'\cdot R\mathbf{p})} f(R\mathbf{x}') |detR| dx'_1 dx'_2 \cdots dx'_d \\
&= \int_{\mathbf{R}^d} e^{2\pi i \mathbf{x}'\cdot \mathbf{p}} f(R\mathbf{x}') dx'_1 dx'_2 \cdots dx'_d \\
&= \widehat{f(R\mathbf{x})}(\mathbf{p}) = \mathcal{F}\mathcal{R}f
\end{aligned}$$

Here in the third line we have used the substitution $\mathbf{x} = R\mathbf{x}'$. □

A radial function will be a function on $\mathbf{R}^d$ that only depends on the distance to the origin, so is invariant under rotations:

Definition. *A radial function is a function satisfying*

$$\mathcal{R}f = f$$

for all rotations R.

The commutativity of $\mathcal{R}$ and $\mathcal{F}$ imply

Claim. *The Fourier transform of a radial function is radial.*

Proof. If f is radial, then $\mathcal{R}f = f$ and by commutativity of $\mathcal{R}$ and $\mathcal{F}$

$$\mathcal{R}\mathcal{F}f = \mathcal{F}\mathcal{R}f = \mathcal{F}f$$

□

In $d = 1$, $O(1)$ is the two element group $\mathbf{Z}_2 = \{I, -I\}$, with one element taking $x \to x$ and the other taking $x \to -x$. In this case a radial function is just an even function, and we have seen previously that the Fourier transform of an even function is even.

3.1 Two dimensions

In two dimensions one has

$$O(2) = SO(2) \times \mathbf{Z}_2$$

meaning that any rotation R can be decomposed into the product of

- An element R_θ in the subgroup $SO(2)$ given by clockwise rotations by angles θ.
- An element of the two element group $\mathbf{Z}_2$ with one element the identity, the other a reflection about an axis, e.g. the reflection $R_- : (x_1, x_2) \to (x_1, -x_2)$ about the x_1 axis.

Two different ways you can work with rotations in two dimensions are

- Identify $\mathbf{R}^2 = \mathbf{C}$ using $z = x_1 + ix_2$. Then elements of the $SO(2)$ subgroup act by
$$z \to R_\theta z = e^{i\theta} z$$
and the reflection R_- acts by conjugation ($z \to \overline{z}$).
- Using matrices, the action of an element of $SO(2)$ is given by
$$R_\theta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
and the action of reflection is
$$R_- \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
One can characterize the rotation matrices as all two by two matrices M that satisfy the condition $MM^T = I$ (here M^T is the transpose of M). One can check that this is the condition on matrices corresponding to the condition that as an action on vectors they preserve the inner product. Since the determinants will satisfy
$$\det(MM^T) = \det M^2 = 1$$
one has $\det M = \pm 1$. The group of these matrices breaks up into a component with determinant 1 (this is $SO(2)$) and a component with determinant -1 (these are a product of an element of $SO(2)$ and a reflection).

To study radial functions in two dimensions, it is convenient to change to polar coordinates, and for a radial function write

$$f(x_1, x_2) = f(r, \theta) = f(r) = f(|\mathbf{x}|)$$

For its Fourier transform (which will also be radial) write

$$\widehat{f}(p_1, p_2) = \widehat{f}(|\mathbf{p}|)$$

If we compute this Fourier transform we find (using the fact that $\mathbf{p}\cdot\mathbf{x} = |\mathbf{p}|r\cos\theta$, where θ is the angle between $\mathbf{x}$ and $\mathbf{p}$)

$$\begin{aligned}\widehat{f}(|\mathbf{p}|) &= \int_{-\pi}^{\pi}\int_{0}^{\infty} f(r)e^{-2\pi i|\mathbf{p}|pr\cos\theta} r dr d\theta \\ &= \int_{0}^{\infty} f(r)r\left(\int_{-\pi}^{\pi} e^{2\pi i|\mathbf{p}|r\sin\theta} d\theta\right) dr\end{aligned}$$

Here we have used $-\cos\theta = \sin(\theta - \frac{\pi}{2})$.

The function in parentheses is a Bessel function, often written

$$J_0(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ir\sin\theta} d\theta$$

More generally, $J_n(r)$ is the n'th Fourier coefficient of the function $e^{ir\sin\theta}$ so

$$J_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ir\sin\theta}e^{-in\theta} d\theta$$

We see that in two dimensions the Fourier transform of a radial function f can be written as

$$\widehat{f}(|\mathbf{p}|) = 2\pi\int_{0}^{\infty} rf(r)J_0(2\pi|\mathbf{p}|r)dr$$

3.2 Three dimensions

In any number of dimensions, rotations are the linear transformations that, written as matrices M, satisfy the condition

$$M^TM = I$$

where M^T is the transpose matrix. To see that this is the same as the condition of preserving the inner product (the dot product), note that

$$\mathbf{x}\cdot\mathbf{y} = \begin{pmatrix}x_1, \cdots, x_d\end{pmatrix}\begin{pmatrix}y_1\\ \vdots \\ y_d\end{pmatrix}$$

and

$$M\mathbf{x}\cdot M\mathbf{y} = \begin{pmatrix}x_1 & \cdots & x_d\end{pmatrix} M^TM\begin{pmatrix}y_1\\ \vdots \\ y_d\end{pmatrix}$$

As noted in the last section $M^TM = I$ implies that $\det M = \pm 1$. The group $O(d)$ breaks up into two components: a subgroup $SO(d)$ of orientation preserving rotations (those with determinant $+1$ and a component of rotations that change orientation, those with determinant -1. If you think of the matrix M as a collection of d column vectors, the condition $M^TM = I$ says that

- Since the off-diagonal elements of I are zero, the dot product of two different column vectors is zero, so they are all pair-wise orthogonal.

- Since the diagonal elements of I are 1, the dot product of each column vector with itself is 1, so the row vectors are not just orthogonal, but orthonormal.

Elements of $O(d)$ can thus be characterized by a set of d orthonormal vectors in $\mathbf{R}^d$, this is just an orthonormal basis in R^d.

In the case $d = 2$, elements of $O(2)$ are given by choosing for the first column a vector on the unit circle in $\mathbf{R}^2$

$$\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

and for the second column one of the two perpendicular unit vectors

$$\pm\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

The positive sign gives determinant $+1$ and thus elements of $SO(2)$, the negative sign gives the other component of $O(2)$. In the case $d = 3$, explicit parametrizations are more complicated, but you can construct all elements of $O(3)$ by

- Pick a first unit vector in $\mathbf{R}^3$. These lie on the unit sphere $S^2 \subset \mathbf{R}^3$ and can be parametrized by two angles.

- Pick a second unit vector in $\mathbf{R}^3$, perpendicular to the first. If you take the first unit vector to point to the North pole of S^2, this second one will lie on a circle S^1 that is the equator of the sphere. These are parametrized by a third angle.

- Pick a third unit vector in $\mathbf{R}^3$, perpendicular to each of the first two. There are two possible choices with opposite sign, one of which will give determinant 1 and an element of $SO(3)$, the other will give determinant -1 and an element of the other component of $O(3)$.

We see that elements of $SO(3)$ can be parametrized by three angles, and are given by a choice of an element of S^2 and an element of S^1. It however is not true that $SO(3) = S^2 \times S^1$ as a space since there is a subtlety: the S^1 is not fixed, but depends on your first choice of an element of S^2.

A standard way to explicitly parametrize elements of $SO(3)$ is by three Euler angles ϕ, θ, ψ, writing a rotation matrix as

$$\begin{pmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}$$

This product of matrices corresponds to (read from right to left) the composition of

- A rotation by angle φ about the 3-axis.
- A rotation by angle θ about the 1-axis.
- A rotation by angle ψ about the (new) 3-axis.

To do calculations in $\mathbf{R}^2$ for which one wants to exploit rotational symmetry, one uses polar coordinates. For problems in $\mathbf{R}^3$ involving rotational symmetry, the standard choice is spherical coordinates r, θ, φ for a vector, where

- r is the distance to the origin.
- θ is the angle between the 3-axis and the vector.
- φ is the angle between the 1-axis and the projection of the vector to the $1-2$ plane.

Should really include a picture here...

The relation between x_1, x_2, x_3 and r, θ, φ coordinates is given by

$$\begin{aligned} x_1 &= r \sin\theta \cos\varphi \\ x_2 &= r \sin\theta \sin\varphi \\ x_3 &= r \cos\theta \end{aligned}$$

Recall that integrals in spherical coordinates are given by

$$\int_{\mathbf{R}^3} F(x_1, x_2, x_3) dx_1 dx_2 dx_3 = \int_0^\infty \int_0^\pi \int_0^{2\pi} F(r, \theta, \varphi) r^2 \sin\theta d\varphi d\theta dr$$

where the range of integration is a full 2π for φ, and half as large (0 to π) for θ. Doing the integral for the Fourier transform in spherical coordinates, for the case of a radial function $f(r, \theta, \varphi) = f(r)$, $\widehat{f}$ will be radial also, only depending on $|\mathbf{p}|$

$$\widehat{f}(|\mathbf{p}|) = \int_0^\infty f(r) \left(\int_0^\pi \int_0^{2\pi} e^{-2\pi i \mathbf{p}\cdot\mathbf{x}} \sin\theta d\varphi d\theta \right) r^2 dr$$

To evaluate the integral in parentheses, use the fact that it is independent of the direction of $\mathbf{p}$, so you might as well take $\mathbf{p}$ in the 3-direction

$$\mathbf{p} = |\mathbf{p}|\mathbf{e}_3$$

If we denote the unit vector in the $\mathbf{x}$ direction by $\widehat{\mathbf{r}}$, then

$$\mathbf{x} = r\widehat{\mathbf{r}}$$

and

$$\mathbf{p} \cdot \mathbf{x} = r|\mathbf{p}| \cos\theta$$

The integral in parentheses then becomes

$$\begin{aligned}\int_0^{\pi}\int_0^{2\pi} e^{-2\pi i r|\mathbf{p}|\cos\theta}\sin\theta d\varphi d\theta =&2\pi\int_0^{\pi} e^{-2\pi i r|\mathbf{p}|\cos\theta}\sin\theta d\theta\\ =&2\pi\int_{-1}^{1} e^{2\pi i r|\mathbf{p}|u}du\\ =&2\pi\frac{1}{2\pi i r|\mathbf{p}|}e^{2\pi i r|\mathbf{p}|u}\Big|_{-1}^{1}\\ =&\frac{2}{r|\mathbf{p}|}\sin(2\pi r|\mathbf{p}|)\end{aligned}$$

(in the second step the substitution is $u = -\cos\theta$). Our final formula for the Fourier transform of a radial function in three dimensions is

$$\widehat{f}(|\mathbf{p}|) = \frac{2}{\mathbf{p}|}\int_0^{\infty} f(r)\sin(2\pi r|\mathbf{p}|)rdr$$

References

[1] Stein, Elias M. and Shakarchi, Rami, Fourier Analysis: An Introduction. Princeton University Press, 2003.