

TOPICS IN REPRESENTATION THEORY: CULTURAL BACKGROUND

This semester we will be covering various topics in representation theory, see the separate syllabus for a detailed list of topics, including some that we may or may not get to. Much of the material will be rather standard for a course of this kind, but an attempt will be made to explain some of the relations between representation theory and quantum mechanics. The notion of “quantization” has motivated several different approaches to the construction of group representations and some of the most important applications of representation theory are to the solution of quantum mechanical problems.

Some important things we won’t be covering include the representation theory of non-compact Lie groups such as $SL(2, \mathbf{R})$, and of arithmetically interesting groups such as $GL(n, \mathbf{Q}_p)$. Our approach to representation theory will be quite geometric, but in the context of differential geometry over the real numbers, not that of algebraic geometry over more general fields, i.e. the theory of “algebraic groups”.

1 Some History

It often helps to have some idea of the history of a field while trying to learn it, so we’ll begin with a short outline of the history of important ideas in the subject, as well as the names of the mathematicians most responsible for their development. For more details of this history, two excellent recent references are [1] and [5].

- 1873: Lie groups. Sophus Lie (1842-1899)
- 1888: Classification of Lie algebras. Wilhelm Killing (1847-1923)
- 1896: Representations of finite groups, characters. Georg Frobenius (1849-1917)
- 1897: Integration over compact Lie groups. Adolf Hurwitz (1859-1919)
- 1905: Schur’s lemma. Issai Schur (1875-1941)
- 1913: Highest weight representations of Lie algebras. Elié Cartan (1869-1951)
- 1925: General representation theory of compact Lie groups, Weyl character formula. Hermann Weyl (1885-1955)
- 1925-6: Quantum mechanics. Werner Heisenberg (1901-1976), Erwin Schrödinger (1887-1961), P. A. M. Dirac (1902-1984)
- 1926: Peter-Weyl theorem.

- 1931: Heisenberg algebra and group, Stone-von Neumann theorem. Weyl, Marshall Stone (1903-1989), John von Neumann (1903-1957)
- 1935-38: Clifford algebras and spinors. Richard Brauer (1901-1977), Weyl, Cartan
- 1951: Representations of non-compact semi-simple groups. Harish-Chandra (1923-1983)
- 1954: Borel-Weil theorem. Armand Borel (1923-), André Weil (1906-1998)
- 1957: Borel-Weil-Bott theorem. Raoul Bott (1923-)
- 1964: Metaplectic Representation. Weil
- 1968: Kac-Moody Lie algebras. Victor Kac (1943-), Robert Moody (1941-)
- 1970: Geometric quantization, method of orbits. Bertram Kostant (1928-), Alexandre Kirillov (1941-)
- 1974 Highest weight representations of Kac-Moody algebras. Kac

2 Quantum Mechanics

The subjects of quantum mechanics and representation theory are closely related in several different ways that I hope to explain throughout this course. Quantum mechanics plays two main roles in representation theory:

- As a source of new constructions of representations, not previously considered by mathematicians.
- As a source of examples and applications of representation theory developed by mathematicians.

The information exchange between mathematicians and physicists in this field over the last 75 years has very much been a two-way street.

We'll begin with a short outline of the relationship between quantum mechanics and representation theory. While having studied quantum mechanics will help understanding this material, this is not necessary since we'll only be using some very simple ideas from the subject and will cover those in this section. There are a lot of readable books of various kinds about quantum mechanics. One of the founders of the subject, Dirac wrote a still useful one [2], a recent one that is aimed at mathematicians is [4].

The fundamental postulates of quantum mechanics are:

1. The state $\Psi(t)$ of a physical system at a given time is given by a vector in a complex Hilbert space \mathcal{H} . Recall that a Hilbert space is a vector space with an inner product $\langle \cdot, \cdot \rangle$, complete with respect to the metric defined by the inner product. Hilbert spaces may be finite or infinite dimensional as complex vector spaces. States are normalized so that $\langle \Psi, \Psi \rangle = 1$.
2. Observable quantities correspond to self-adjoint operators A on \mathcal{H} . The observable quantity corresponding to A will take the value a for states Ψ_a that are eigenvectors of A with eigenvalue a (i.e. $A\Psi_a = a\Psi_a$).
3. The most important observable is the energy and its corresponding self-adjoint operator is the Hamiltonian H . The time evolution of a system is given by the Schrödinger equation

$$i\hbar \frac{d}{dt} \Psi(t) = H\Psi(t)$$

where \hbar is a constant (Planck's constant divided by 2π). We can choose our energy and/or time units so that $\hbar = 1$ and will generally do so.

Some comments on notation and conventions:

Physicists like to write vectors in \mathcal{H} as “kets” $|\Psi\rangle$, with elements of the dual space written as “bras” $\langle\Psi|$, and the inner product written as

$$\langle\Psi', \Psi\rangle = \langle\Psi'|\Psi\rangle$$

There are two different conventions for the standard hermitian inner product on \mathbf{C}^n . Traditionally mathematicians have often preferred

$$(z', z) = \sum_{i=1}^n z'_i \bar{z}_i$$

while physicists are uniformly in favor of

$$(z', z) = \sum_{i=1}^n \bar{z}'_i z_i$$

Since Simon and Segal use the latter, I'll follow them and the physicists.

The Schrödinger equation says that $i\frac{d}{dt}$ is a self-adjoint operator (the Hamiltonian H), in which case $\frac{d}{dt}$ is the skew-adjoint operator $-iH$. Recall that, in finite dimensions, for an n -dimensional complex vector space with the standard Hermitian inner product, the skew-adjoint linear operators form the Lie algebra $\mathfrak{u}(n)$. While mathematicians like to think of the Lie algebra of the unitary group as consisting of skew-adjoint operators, physicists like to multiply by i and work with self-adjoint operators.

The solution to the Schrödinger equation is formally given by

$$\Psi(t) = U(t)\Psi(0), \quad U(t) = e^{-iHt}$$

(we've set $\hbar = 1$). Self-adjointness of H implies that $U(t)$ is unitary, so (Ψ, Ψ) is time-independent (and can thus be consistently set to 1 for all times). For this formal solution to make sense we just need to be able to make sense of the exponential and show that it has the expected property

$$U(t_1)U(t_2) = U(t_1 + t_2)$$

For a bounded operator Hamiltonian operator H this is easy, you can just use the power series definition of the exponential. For unbounded operators some more serious analysis is required.

One can reformulate this discussion as saying that

$$t \rightarrow e^{-iHt}$$

is a continuous unitary representation of the additive group \mathbf{R} on the Hilbert space \mathcal{H} . Stone's theorem (1930) tells us that there is a one-to-one correspondence between such representations and self-adjoint operators. This is actually a classification theorem for representations of \mathbf{R} . It shows that understanding the representations of even the simplest non-compact Lie group is equivalent to an intractable problem in analysis, that of understanding all possible self-adjoint operators on Hilbert space. This is one of the main reasons we will mostly be restricting our attention to representations of compact Lie groups this semester.

In the above situation the physical system is invariant under translations of the time variable (the Hamiltonian is time-independent), and this group \mathbf{R} of time translations is represented on \mathcal{H} . The Hamiltonian operator is said to be a "generator" of the \mathbf{R} symmetry.

Besides time translation symmetry, physical systems generally have other symmetries, examples are:

- Translations in spatial directions.
- Rotations in space.
- Lorentz transformations in special relativity.
- Phase transformations of the wave-function.
- $U(N)$ transformations amongst N different kinds of particles. ("colors" or "flavors").
- S_n permutation transformations amongst n identical particles.

The Hilbert space of a quantum mechanical system will carry a unitary representation of any such symmetry groups of the physical system. Thus quantum mechanics produces interesting representations of all these groups and a sizable part of understanding the quantum mechanics of a physical system comes down to seeing how its Hilbert space decomposes into irreducible representations of these symmetry groups.

Since the eigenvectors and eigenvalues of all operators are unchanged when one multiplies all vectors in \mathcal{H} by an overall phase $e^{i\theta}$, vectors that differ just

by such a phase are physically equivalent. As a result the representations that naturally occur in quantum mechanics are sometimes just “projective representations”. In other words the unitary transformations $U(g)$ of \mathcal{H} satisfy

$$U(g_1)U(g_2) = e^{i\theta(g_1, g_2)}U(g_1g_2)$$

for g_1, g_2 two group elements and some angle $\theta(g_1, g_2)$. Another way of thinking of this is that quantum mechanics often provides representations on the projective space $P(\mathcal{H})$ of complex lines in \mathcal{H} rather than on \mathcal{H} itself. One important example of this phenomenon involves the group $SO(3)$ of spatial rotations. This group is often only represented projectively on \mathcal{H} , although there is a true representation of its double cover $SU(2)$.

How do physicists produce examples of quantum mechanical systems? One method involves starting with a classical mechanical system and trying to produce a corresponding quantum mechanical one. Geometrical methods for doing this go by the name of “geometrical quantization” and we will examine this notion in some detail later on in the course. Two possible places to do some reading about this are [6] and [3].

The traditional formulation of classical “Hamiltonian” mechanics involves a choice of “dynamical variables” which are coordinates

$$(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n)$$

on an even-dimensional “phase space” $P = \mathbf{R}^{2n}$. On this space there is a distinguished function, the Hamiltonian

$$H(q_i, p_i)$$

and a bilinear operation on functions, the “Poisson Bracket”

$$\{f, g\}_{PB} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right)$$

The time evolution of any function $u(q_i, p_i)$ of the dynamical variables is given by

$$\frac{du}{dt} = \{u, H\}_{PB}$$

in particular for $u = q_i$ and $u = p_i$ we have Hamilton’s equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

This formulation is very much coordinate dependent. We can adopt a coordinate invariant language and at the same time allow phase spaces P that are only locally \mathbf{R}^{2n} by defining a classical mechanical system as consisting of a symplectic manifold P and a Hamiltonian function

$$H : P \rightarrow \mathbf{R}$$

. A symplectic manifold P is an even dimensional manifold equipped with a “symplectic form”, this is a two form $\omega \in \Omega^2(P)$ such that:

1. ω is closed, $d\omega = 0$
2. ω is non-degenerate on P (i.e. for a non-zero vector field X on P , $\omega(X, \cdot)$ is an everywhere non-zero one-form).

In the case of $P = \mathbf{R}^{2n}$ above

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i$$

is a symplectic form.

On a Riemannian manifold we can associate to a function f a vector field, the gradient ∇f , by taking the 1-form df and using the metric to get a vector field. On a symplectic manifold we can do something similar using the symplectic 2-form instead of the metric, associating to a function f its “symplectic gradient” X_f , a vector field satisfying

$$df = \omega(X_f, \cdot)$$

The Poisson bracket of two functions is just

$$\{f, g\}_{PB} = \omega(X_f, X_g)$$

and time evolution on P is determined in terms of the Hamiltonian by just integrating the vector field X_H .

For the simplest possible example of all this (the harmonic oscillator), take $P = \mathbf{R}^2$ and $H = q^2 + p^2$. Trajectories are just circles about the origin and the symplectic form is just the area two-form.

A “geometric quantization” is some scheme that should associate to the symplectic manifold (P, ω) a Hilbert space \mathcal{H} . A transformation of (P, ω) that preserves the ω is called a “symplectomorphism” by mathematicians, a “canonical transformation” by physicists. Given a group of such transformations of P , one would like quantization to give a unitary representation of the group on \mathcal{H} . One such group is the group \mathbf{R} given by the flow along the Hamiltonian vector field X_H (time evolution). We have seen that in quantum mechanics the Schrödinger equation implies that this has a continuous unitary representation on \mathcal{H} . In general (P, ω) has a large group of symplectomorphisms and we would like our quantization procedure to produce unitary representations of at least some subgroups of this large group. This turns out to be possible in many interesting cases, but not in general since the Groenwald-van Hove theorem (late 40s) shows there isn’t a consistent quantization map producing a unitary representation of the entire group.

References

- [1] Borel, A., *Essays in the History of Lie Groups and Algebraic Groups*, American Mathematical Society, 2001.

- [2] Dirac, P. A. M., *The Principles of Quantum Mechanics (Fourth Edition)*, Oxford University Press, 1958.
- [3] Guillemin, V., and Sternberg, S., *Symplectic Techniques in Physics*, Cambridge University Press, 1984.
- [4] Hannabuss, K., *An Introduction to Quantum Theory*, Oxford University Press, 1997.
- [5] Hawkins, T., *Emergence of the Theory of Lie Groups*, Springer, 2000.
- [6] Woodhouse, N., *Geometric Quantization*, Oxford University Press, 1992.